

Combinatorial Seifert fibred spaces with transitive cyclic automorphism group

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Abstract

In combinatorial topology we aim to triangulate manifolds such that their topological properties are reflected in the combinatorial structure of their description. Here, we give a combinatorial criterion on when exactly triangulations of 3-manifolds with transitive cyclic symmetry can be generalised to an infinite family of such triangulations with similarly strong combinatorial properties.

In particular, we construct triangulations of Seifert fibred spaces with transitive cyclic symmetry where the symmetry preserves the fibres and acts non-trivially on the homology of the spaces. The triangulations include the Brieskorn homology spheres $\Sigma(p, q, r)$, the lens spaces $L(q, 1)$ and, as a limit case, $(\mathbf{S}^2 \times \mathbf{S}^1)^{\#(p-1)(q-1)}$.

MSC 2010: 57Q15; 57N10; 05B10; 20B25;

Keywords: combinatorial topology, (transitive) combinatorial 3-manifold, Seifert fibred space, Brieskorn homology sphere, (transitive) permutation group, difference cycle, cyclic 4-polytope, cyclic group

1 Introduction

It is the defining goal of combinatorial topology to establish links between the combinatorial structure of an object and its topology. Of course, this is not possible in general since each individual topological object can usually be described by a large and diverse class of different combinatorial objects, typically with very distinct properties. Hence the question of how to choose a combinatorial structure which describes a topological object “best” is of critical importance.

If the right constraints are imposed on the combinatorial structure of an object, topological properties become transparent which otherwise are hard to obtain. For instance a simplicial complex where every triple of vertices spans a triangle has to be simply connected [15].

In other words, the right choice of combinatorial object makes the topology of a manifold combinatorially accessible.

In “non-combinatorial” (conventional) 3-manifold topology there are well established methods for describing manifolds in ways that make their topological structure easily understandable. One of these methods makes use of the fact that any closed oriented 3-manifold can be obtained from the 3-sphere by repeatedly applying *Dehn surgery*. Moreover, there is the standard JSJ decomposition [12, 13] for prime 3-manifolds where *Seifert fibred spaces* come naturally out of the construction. Seifert fibred spaces are 3-manifolds which are obtained by starting with a very restricted and well understood class of fibrations of the circle over a surface, followed by performing surgery parallel to the fibres.

In the combinatorial setting we work with *combinatorial manifolds* which are simplicial complexes with some additional properties. As a result even the basic form of Dehn surgery needed to construct Seifert fibred spaces introduces unwanted complexity because gluing simplicial complexes together can require significant and sometimes unwieldy modifications. The GAP-script SEIFERT [20] by Lutz and Brehm constructs arbitrary combinatorial *Seifert fibred spaces*. However, due to the added complexity of the gluings involved, the output complexes of the script are typically difficult to analyse.

In this article we aim to overcome this difficulty by explicitly constructing combinatorial structures that reflect the topological properties of the objects we want to represent. More precisely, since Seifert fibrations are unions of disjoint circles, we focus on combinatorial 3-manifolds which, in a certain sense, are invariant under rotations. In more combinatorial terms, we are interested in complexes with *transitive cyclic symmetries*; that is, complexes with automorphism groups acting transitively on its vertices.

In addition to the philosophical compatibility of a rotational symmetry with \mathbf{S}^1 -fibrations, combinatorial 3-manifolds with transitive cyclic symmetry have a number of other appealing properties. They are globally determined by only a local neighbourhood, which means that the amount of data needed to describe them is much smaller than the complex itself. Furthermore, they are easy to construct due to their transitive symmetry, and particularly easy to analyse due to the simplicity of the cyclic group. As a consequence, this type of combinatorial manifold has been a canonical choice for a good representative of the underlying topological manifold in the work of many authors over the past decades (for instance, see [4, 16, 21, 22, 26]).

In addition to constructing such combinatorial manifolds we are interested in making these constructions compatible with Dehn surgery. Of course, working exclusively with combinatorial 3-manifolds with vertex transitive cyclic automorphism group implies even stronger restrictions to performing Dehn surgery than the restrictions already present in the general combinatorial setting. As a consequence, despite all research about combinatorial manifolds with transitive symmetry, there are only very few examples of combinatorial surgery preserving a given transitive cyclic symmetry.

- There is a 14-vertex triangulation of the 3-sphere containing two disjoint solid 7-vertex tori in form of one *difference cycle*, i.e., an orbit of the action of the transitive cyclic automorphism group on the triangulation. This difference cycle can be replaced by another difference cycle with equal boundary yielding a triangulation of $\mathbf{S}^2 \times \mathbf{S}^1$ and, in a slightly different setting, a triangulation of the lens space $L(3, 1)$ [25, Section 4.5.1].
- In [16] Kühnel and Lassmann construct an infinite family of neighbourly 3-dimensional combinatorial n -vertex Klein bottles, $n \geq 9$, using a special property of the boundary complex of the *cyclic 4-polytope* $C_4(n)$: By *Gale's evenness condition*, the boundary of the 4-dimensional cyclic polytope with n vertices $\partial C_4(n)$, $n \geq 9$, can be decomposed into two n -vertex solid tori $A(n)$ and $B(n)$. This yields a *handlebody decomposition of genus one* of the combinatorial 3-sphere $\partial C_4(n)$ respecting the transitive cyclic symmetry (cf. for example [15, Section 5B]) and hence provides an excellent starting point to perform Dehn surgery in a combinatorial setting with transitive symmetry.
- In [26] a related technique is used to construct a family of infinitely many distinct lens spaces L_k : For every $k \geq 0$, a $14 + 4k$ vertex base complex is glued to two solid tori, this way realising combinatorial surgery in infinitely many distinct ways.

We want to exploit the above constructions, and in particular the decomposition of $\partial C_4(n)$, to build Seifert fibred spaces where the combinatorics of the complex reflects the topological structure of the fibration (i.e., combinatorial Seifert fibred spaces with transitive cyclic symmetry in which solid tori such as $A(n)$ and $B(n)$ can be plugged in to build neighbourhoods of the exceptional fibres).

For example, in the genus one handlebody decomposition of the boundary complex of the cyclic 4-polytope $\partial C_4(n) = A(n) \cup B(n)$ we replace $A(n)$ by another simplicial complex $\tilde{A}(n)$ with transitive cyclic symmetry and equal boundary. This gives rise to a closed complex in which $B(n)$ acts as an embedded solid torus where the gluing map depends on the number of vertices n and the choice of a particular decomposition $\partial C_4(n) = A(n) \cup B(n)$ (cf. parameter l in Equations (2.2) and (2.3)).

Constructing such complexes is not trivial in general, but strongly depends on one of the key properties of combinatorial manifolds with transitive symmetry: these complexes are easy to find.

In [26] there is a classification of combinatorial 3-manifolds with transitive cyclic symmetry up to 22 vertices. Searching this classification for complexes containing a solid torus of type $B(n_0)$ (for a fixed $n_0 \leq 24$) resulted in a large number of candidates for families of Seifert fibred spaces (the complete list is available from the second author upon request).

Our first main result Theorem 1.1 essentially describes a setting where a single example of “combinatorial surgery” can be *expanded* into an infinite family of such examples. Using Theorem 1.1, the candidates above can then be checked for whether or not they allow such an expansion to an infinite family of combinatorial 3-manifolds and hence into a candidate for a *family* of Seifert fibred spaces as described above.

Theorem 1.1. *Let M be an n -vertex combinatorial 3-manifold, n even, given by m difference cycles $d_1, 1 \leq i \leq m$ and $(1 : n/2 - 1 : 1n/2 - 1)$. Then for all $k \geq 0$, M admits an expanded version M_k with $n + k$ vertices if and only if each difference cycle d_i contains an entry greater or equal to $n/2$. If M is neighbourly M_k is neighbourly and vice versa for all $k \geq 0$.*

The above construction is made more precise and explained in detail in Section 3.

Theorem 1.1 describes families of combinatorial 3-manifolds with transitive cyclic symmetry. In the course of this article we show that this construction is suitable to find expansions of triangulated Seifert fibred spaces with multiple exceptional fibres where different levels of expansion, i.e., different values of k in the above description, determine different types of exceptional fibres. This provides a more systematic approach for describing combinatorial surgeries like the ones mentioned above (cf. [16, 25, 26]) and allow more complex constructions. In particular, we present the following 3-parameter family of triangulations of Seifert fibred spaces with an unbounded number of exceptional fibres.

Theorem 1.2. *There is a 3-parameter family $M(p, q, r)$, $2 \leq p < q$ co-prime, $r > 0$, of combinatorial Seifert fibred spaces with $2pq + r$ vertices and transitive cyclic automorphism group of topological type*

$$\text{SFS}[(\mathbb{T}^2)^{\#(a-1)(b-1)/2} : (-p/a, b_1)^b (q/b, b_2)^a (-r/ab, b_3)]$$

where $(\mathbb{T}^2)^{\#g}$ is the orientable surface of genus g , $(x, y)^\nu$ denotes a set of ν exceptional fibres of type (x, y) , $a := \gcd(p, r)$, $b := \gcd(q, r)$, and

$$\left(\frac{b_1}{p} - \frac{b_2}{q} + \frac{b_3}{r}\right) \frac{pqr}{ab} = 1.$$

The isomorphism type of the Seifert fibration is determined by these conditions and, in particular, we have

- (i) $M(p, q, r)$ is the Brieskorn homology sphere $\Sigma(p, q, r)$ whenever p, q and r are co-prime,
- (ii) $M(2, q, 2)$ is the lens space $L(q, 1)$ and

In the case $r = 0$ we do not obtain Seifert fibred spaces but the manifolds $(\mathbf{S}^2 \times \mathbf{S}^1)^{\#(p-1)(q-1)}$.

We will see that the difference cycles of $M(p, q, r)$ already reveal where the fibres are running within the combinatorial manifold. Moreover, by the transitive cyclic symmetry the analysis of the complexes can be done by only considering a fraction of the actual complex and with the help of the tools of design theory.

The nice combinatorial structure of the complexes allow us to deduce further topological properties of the Seifert fibred spaces. Namely, we can show the following two results.

Theorem 1.3. *$M(p, q, r)$ is of Heegaard genus at most $(p - 1)(q - 1)$.*

Theorem 1.4. *The automorphism group*

$$G := \text{Aut}(M(2, q, 2kq)) \cong \mathbb{Z}_{2q(k+2)},$$

q prime, $k \geq 0$, acts on the first homology group $H_1(M(2, q, 2kq), \mathbb{Z}) = \mathbb{Z}^{q-1}$ by

$$\rho: G \rightarrow \text{SL}(q-1, \mathbb{Z}); \quad g \mapsto \begin{pmatrix} 0 & \cdots & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}^g$$

where $|\rho(G)| = 2q$.

For p and q fixed Theorem 1.3 gives us infinite families of 3-manifolds of bounded Heegaard genus. This is interesting, as bounds for the Heegaard genus of a 3-manifold are usually hard to obtain in a purely combinatorial setting. Moreover, we show that this bound is tight whenever $r \equiv 0 \pmod{pq}$ and for $(p, q, r) = (2, 3, \geq 3)$.

Theorem 1.4 describes an interplay between the automorphism group of $M(2, q, 2kq)$ for q prime (a combinatorial object) and its first homology group (a topological invariant). Intuitively, a combinatorial manifold should be presented in a way such that any symmetry of the combinatorial structure is meaningful for the topological object. For example for the d -dimensional torus

$$\mathbb{T}^d = \mathbf{S}^1 \times \dots \times \mathbf{S}^1$$

we would like to have a triangulation where each symmetry of the combinatorial object permutes the \mathbf{S}^1 -components, for a connected sum of manifolds

$$M^{\#k} = M \# \dots \# M$$

we would like the symmetries to act on the direct summands, and so on.

In more general terms, if for a combinatorial manifold M the first homology group $H_1(M, R)$ is a free R -module of rank k , we would like to have a non-trivial representation of the automorphism group $\text{Aut}(M)$ of the form

$$\rho: \text{Aut}(M) \rightarrow \text{SL}(k, R). \quad (1.1)$$

However, as of today, few examples are known where such a non-trivial representation exist. Theorem 1.4 describes an infinite family of further examples using the complex $M(2, q, 2kq)$ in the case that q is a prime.

Finally, there are many more interesting families of Seifert fibred spaces and using Theorem 1.1 more can be found. However, the question whether or not this construction principle is suitable to obtain a significant proportion of *all* Seifert fibred spaces with a similar degree of impact of the topology on the combinatorics is unanswered as of today and subject to work in progress.

2 Preliminaries

2.1 3-manifolds and Seifert fibred spaces

By work of Moise [23] it follows that every topological 3-manifold admits a unique piecewise linear and smooth structure and hence all 3-dimensional manifolds can be triangulated. As a corollary, it follows that every 3-manifold M can be decomposed into two *handlebodies*, i.e., thickened graphs, which are joined along their boundary surface in order to give M . The genus of the boundary surface is said to be the *genus of the handlebody decomposition of M* and the minimum genus over all handlebody decompositions of M is called the *Heegaard genus* of M . A modification of this construction results in the observation that every 3-manifold M can be constructed from the

3-sphere, by drilling out solid tori and gluing them back such that the meridian of the old solid torus in M is identified with a torus knot of type (p, q) on the boundary of the new solid torus. Such a drilling operation is called *Dehn surgery* of type (p, q) (see [19, Theorem 12.14] for more about Dehn surgery).

3-manifolds can be uniquely decomposed into a connected sum of so-called *prime 3-manifolds* which cannot be represented as a non-trivial connected sum. One important class of prime 3-manifolds can be described as a fibration of the circle over a 2-dimensional base orbifold with a finite number of additional Dehn surgeries performed along thickened fibres (note that a thickened fibre is a solid torus). Such a representation is called a *Seifert fibred space* and is determined by the base surface, the type of the fibration and a list of (rational) Dehn surgeries along the fibres each specified by a pair of co-prime integers (see [24] for more about Seifert fibrations).

2.2 Combinatorial manifolds

We can represent manifolds in a purely combinatorial piecewise linear fashion using simplicial complexes. For each vertex v in a simplicial complex C we refer to the *link* of v as the boundary of its simplicial neighbourhood, i.e., in the set of all simplices containing v the set of proper faces not containing v . A *combinatorial d -manifold* is a pure and abstract d -dimensional simplicial complex such that each vertex link is a triangulated $(d - 1)$ -sphere with the standard piecewise linear structure. If, in a simplicial complex, the link of a vertex v is *not* a triangulated $(d - 1)$ -sphere with the standard piecewise linear structure, v is referred to as a *singular vertex*. A combinatorial d -manifold is said to be *neighbourly*, if the underlying simplicial complex contains all possible $\binom{n}{2}$ edges where n is the number of vertices. A combinatorial d -manifold always has a piecewise linear structure induced by the simplicial complex. In general, this is not true for simplicial complexes homeomorphic to a manifold (so-called *triangulations of manifolds*) as illustrated by a triangulation of Edward's sphere in dimension 5 in [3]. Hence using the notion of a combinatorial manifold is necessary if we want to work with piecewise linear manifolds.

However, in dimension 3 things are simpler – any two triangulations of the same 3-manifold are equivalent and induce a unique piecewise linear structure by Moise's theorem [23] (cf. above), and every triangulated 3-manifold is automatically a combinatorial 3-manifold.

In the following sections, we refer to combinatorial 3-manifolds which are homeomorphic to Seifert fibred spaces as *combinatorial Seifert fibred spaces*.

2.3 Automorphism groups and difference cycles

Any abstract simplicial complex and hence any combinatorial manifold M can be seen as a combinatorial structure consisting of tuples of elements of $V = \{0, 1, \dots, n - 1\}$ where each element of V appears in at least one tuple. The elements of V are referred to as the vertices of the complex.

The *automorphism group* $\text{Aut}(M)$ of M is the group of all permutations $\sigma \in S_n$ of the vertices of M which do not change the complex M as a whole. If $\text{Aut}(M)$ acts transitively on the vertices, i.e., if for any pair of vertices u and v there is an automorphism $\sigma \in \text{Aut}(M)$ such that $\sigma \cdot u = v$, M is called a *combinatorial manifold with transitive automorphism group* or just a *transitive combinatorial manifold*. If a transitive combinatorial manifold is invariant under the cyclic \mathbb{Z}_n -action $v \mapsto v + 1 \pmod n$ (i.e., if for a combinatorial manifold M , possibly after a relabelling of the vertices, $\mathbb{Z}_n = \langle (0, 1, \dots, n - 1) \rangle$ is a subgroup of $\text{Aut}(M)$), then M is called a *cyclic combinatorial manifold* (here $\langle (a, b, c, \dots) \rangle$ denotes the permutation group generated by the permutation (a, b, c, \dots) given in cycle notation).

For cyclic combinatorial manifolds we have the following special situation: Since the entire complex does not change under a vertex-shift of type $v \mapsto v + 1 \pmod n$, two tuples are in one orbit of the cyclic group action if and only if the differences modulo n of its vertices are equal. Hence we can compute a system of orbit representatives by just looking at the differences modulo n of the vertices of all tuples of the combinatorial manifold (cf. [17]). This motivates the following definition.

Definition 1 (Difference cycle). Let a_i , $0 \leq i \leq d$, be positive integers, $n := \sum_{i=0}^d a_i$ and $\mathbb{Z}_n = \langle (0, 1, \dots, n-1) \rangle$. The simplicial complex

$$(a_0 : \dots : a_d) := \mathbb{Z}_n \cdot \langle 0, a_0, \dots, \sum_{i=0}^{d-1} a_i \rangle$$

is called a *difference cycle of dimension d on n vertices* where $\mathbb{Z}_n \cdot \langle \cdot \rangle$ denotes the \mathbb{Z}_n -orbit of $\langle \cdot \rangle$. The number of elements of $(a_0 : \dots : a_d)$ is referred to as the *length* of the difference cycle. If a simplicial complex C is a union of difference cycles of dimension d on n vertices and λ is a unit of \mathbb{Z}_n such that the complex λC (obtained by multiplying all vertex labels by λ modulo n) equals C , then λ is called a *multiplier* of C .

Note that for any unit $\lambda \in \mathbb{Z}_n^\times$, the complex λC is combinatorially isomorphic to C . In particular, all $\lambda \in \mathbb{Z}_n^\times$ are multipliers of the complex $\bigcup_{\lambda \in \mathbb{Z}_n^\times} \lambda C$ by construction. The definition of a difference cycle above is equivalent to the one given in [17].

Throughout this article, we describe *cyclic combinatorial manifolds* as a set of difference cycles with the implication that we take the union of the difference cycles to describe the simplicial complex. In this way, many problems dealing with cyclic combinatorial manifolds can be solved in an elegant way.

2.4 Cyclic polytopes and combinatorial exceptional fibres

The family of *cyclic polytopes* is a two parameter family $C_d(n)$ of convex simplicial d -polytopes given by the convex hull of n distinct points on the momentum curve

$$\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^d; \quad t \mapsto (t, t^2, \dots, t^d)^T.$$

Cyclic polytopes were first described by Carathéodory at the beginning of the 20th century [7] and have played an important role in polytope theory and combinatorics ever since.

A remarkable property of cyclic polytopes is that their combinatorial structure can be described by *Gale's evenness condition* [11]. Labelling the vertices of the polytope $C_d(n)$ by the integers $V(C_d(n)) = \{0, 1, \dots, n-1\}$ for increasing t , a d -tuple $\Delta \subset V(C_d(n))$ is a facet of $C_d(n)$ if and only if all pairs of vertices in the complement $V(C_d(n)) \setminus \Delta$ are separated by an even number of vertices in Δ .

This has the following consequence in even dimensions $2m$. A $2m$ -tuple $\Delta := \langle a_0, \dots, a_{2m-1} \rangle$ is a facet of $C_{2m}(n)$ if and only if $\Delta + l := \langle a_0 + l \bmod n, \dots, a_{2m-1} + l \bmod n \rangle$ is a facet of $C_{2m}(n)$ for all $l \geq 0$. Hence $C_{2m}(n)$ has an automorphism group $\text{Aut}(C_{2m}(n))$ containing $\mathbb{Z}_n = \langle (0, 1, \dots, n-1) \rangle$ as a subgroup acting transitively on the vertices. To see this, shift the labels of Δ and of an arbitrary pair of vertices $\{v_1, v_2\} \subset V(C_{2m}(n)) \setminus \Delta$, $v_1 < v_2$, by $n - v_2$. Since Δ contains an even number of vertices and $\{v_1, v_2\}$ is arbitrary, $\Delta + (n - v_2)$ satisfies Gale's evenness condition if and only if Δ satisfied Gale's evenness condition.

By Gale's evenness condition, the vertex labels of a facet of $C_{2m}(n)$ split into sequences

$$l, (l+1) \bmod n, (l+2) \bmod n, (l+3) \bmod n, \dots$$

of even length. Consequently, a difference cycle D is contained in $C_{2m}(n)$ if and only if D can be written as a concatenation of sequences of consecutive 1-entries of odd length followed by a single difference greater than 1. In the case $2m = 4$, the observations above give rise to the following way to describe $\partial C_4(n)$.

$$\partial C_4(n) := \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} \{(1 : i : 1 : n - 2 - i)\}. \quad (2.1)$$

Note that in Equation (2.1) all 3-dimensional difference cycles consisting of sequences of 1-entries of odd length followed by single entries greater than 1 are listed. From the viewpoint of 3-manifold theory this description reveals another interesting property. By a simple collapsing argument we can see that

$$A(l, n) := \bigcup_{i=1}^l \{(1 : i : 1 : n - 2 - i)\} \quad (2.2)$$

as well as

$$B(l, n) := \bigcup_{i=l+1}^{\lfloor \frac{n}{2} \rfloor} \{(1 : i : 1 : n - 2 - i)\} \quad (2.3)$$

are triangulated solid tori for all $1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor - 1$, thus yielding a *handlebody decomposition of genus one* of the combinatorial 3-sphere $\partial C_4(n)$ respecting its transitive cyclic symmetry (cf. for example [15, Section 5B]). Solid tori like $A(l, n)$, $B(l, n)$ and related constructions provide families of distinct pairs of solid tori with equal boundary and thus provide an excellent set of starting points to perform Dehn surgery in a combinatorial setting. For this reason we refer to them as *combinatorial exceptional fibres*.

2.5 rsl-functions

One of the principal tools to analyse combinatorial manifolds is the use of a discrete Morse type theory following Kuiper, Banchoff and Kühnel [1, 2, 15, 18]. In this theory, the discrete analogue of a Morse function is given by a mapping from the set of vertices V of a combinatorial manifold M to the real numbers \mathbb{R} such that no two vertices have the same image, in this way inducing a total ordering on V . This mapping can then be extended to a function $f : M \rightarrow \mathbb{R}$ by linearly interpolating the values of the vertices of a face of M for all points inside that face. f is called a *regular simplexwise linear function* or *rsl-function* on M .

A point $x \in M$ is said to be *critical* for an rsl-function $f : M \rightarrow \mathbb{R}$ if

$$H_*(M_x, M_x \setminus \{x\}, \mathbb{F}) \neq 0$$

where $M_x := \{y \in M \mid f(y) \leq f(x)\}$ and \mathbb{F} is a field. Here, H_* denotes simplicial homology. It follows that no point of M can be critical except possibly the vertices, also, in contrast to classical Morse theory, a point can be critical of multiple indices and with higher multiplicity. More precisely we call a vertex v *critical of index i and multiplicity m* if $\beta_i(M_v, M_v \setminus \{v\}, \mathbb{F}) = m$.

A result of Kuiper [18] states that the number of critical points of an rsl-function of M counted by multiplicity is an upper bound for the sum of the Betti numbers of M , hence extending the famous Morse relations from the smooth theory to the discrete case. In addition, like in the smooth case the alternating sum over the critical points of index i of *any* rsl-function of M counted by multiplicity equals the alternating sum over the Betti numbers of M and thus the Euler characteristic of M .

3 Proof of Theorem 1.1

Theorem 1.1 gives a purely combinatorial criterion for when a given cyclic combinatorial 3-manifold can be expanded to an infinite family of combinatorial 3-manifolds and hence to a candidate for a family of combinatorial Seifert fibred spaces (of distinct topological types). For similar (but different) results about cyclic combinatorial manifolds see Theorem 3.1 and Theorem 3.7 in [26].

Before we proof Theorem 1.1 we first introduce some notation to make the statement of the theorem more precise: Let $d_i = (d_i^0 : d_i^1 : d_i^2 : d_i^3)$, $1 \leq i \leq m$, be difference cycles with n vertices, n even, where w.l.o.g. $d_i^3 \geq d_i^j$ for all $0 \leq j \leq 2$, $1 \leq i \leq m$, and let $d_{i,k}$, $1 \leq i \leq m$, $k \geq 0$, be difference cycles with $n + k$ vertices given by $d_{i,k} = (d_i^0 : d_i^1 : d_i^2 : d_i^3 + k)$.

Then the n -vertex combinatorial 3-manifold M is given by

$$M = \left\{ d_1, \dots, d_m, \left(1 : \frac{n}{2} - 1 : 1 : \frac{n}{2} - 1 \right) \right\}.$$

Now Theorem 1.1 states that for all $k \geq 0$ the combinatorial manifold M has an $n + k$ -vertex expansion, noted as

$$M_k = \{d_{1,k}, \dots, d_{m,k}\} \bigcup_{\ell=\frac{n}{2}}^{\lfloor \frac{n+k}{2} \rfloor} \{(1 : \ell - 1 : 1 : n - \ell - 1)\},$$

if and only if $d_i^0 + d_i^1 + d_i^2 \leq \frac{n}{2}$ for all $1 \leq i \leq m$.

In addition, given this notation, any combinatorial 3-manifold of the form M_k , that is, $d_i^0 + d_i^1 + d_i^2 + k \leq \frac{n}{2}$ for all $1 \leq i \leq m$, is the k -th member of such an expansion series.

Proof of Theorem 1.1. Let M be a combinatorial 3-manifold with n vertices given by

$$M = \left\{ d_1, \dots, d_m, \left(1 : \frac{n}{2} - 1 : 1 : \frac{n}{2} - 1 \right) \right\},$$

$d = (d_i^0 : d_i^1 : d_i^2 : d_i^3)$ such that $d_i^0 + d_i^1 + d_i^2 \leq \frac{n}{2}$ for all $1 \leq i \leq m$.

Throughout the proof we use the following naming convention. Instead of identifying the n vertices of M with the elements of \mathbb{Z}_n we use the integers $-\frac{n}{2} + 1, -\frac{n}{2} + 2, \dots, \frac{n}{2} - 1$ and $\pm \frac{n}{2}$ (note that n is even) where the labels coincide with the elements of \mathbb{Z}_n when taken modulo n . The tetrahedra containing vertex 0 in M are then given by

$$\begin{aligned} & \bigcup_{i=1}^m \left\{ \langle 0, d_i^0, d_i^0 + d_i^1, d_i^0 + d_i^1 + d_i^2 \rangle, \langle -d_i^0, 0, d_i^1, d_i^1 + d_i^2 \rangle, \right. \\ & \quad \left. \langle -d_i^0 - d_i^1, -d_i^1, 0, d_i^2 \rangle, \langle -d_i^0 - d_i^1 - d_i^2, -d_i^1 - d_i^2, -d_i^2, 0 \rangle \right\} \\ & \cup \left\{ \langle -\frac{n}{2} + 1, 0, 1, \pm \frac{n}{2} \rangle, \langle -1, 0, \frac{n}{2} - 1, \pm \frac{n}{2} \rangle \right\} \end{aligned}$$

In a similar fashion we name the $n + k$ vertices of M_k by $-\frac{n}{2} + 1, -\frac{n}{2} + 2, \dots, \frac{n}{2}, \frac{n}{2} + 1, \dots, \frac{n}{2} + k - 1$ and we identify $-\frac{n}{2} = \frac{n}{2} + k$. Then we have for the tetrahedra containing 0 in M_k

$$\begin{aligned} & \bigcup_{i=1}^m \left\{ \langle 0, d_i^0, d_i^0 + d_i^1, d_i^0 + d_i^1 + d_i^2 \rangle, \langle -d_i^0, 0, d_i^1, d_i^1 + d_i^2 \rangle, \right. \\ & \quad \left. \langle -d_i^0 + d_i^1, -d_i^1, 0, d_i^2 \rangle, \langle -d_i^0 - d_i^1 - d_i^2, -d_i^1 - d_i^2, -d_i^2, 0 \rangle \right\} \\ & \bigcup_{\ell=\frac{n}{2}}^{\lfloor \frac{n+k}{2} \rfloor} \left\{ \langle 0, 1, \ell, \ell + 1 \rangle, \langle -1, 0, \ell - 1, \ell \rangle, \langle n - \ell, n - \ell + 1, 0, 1 \rangle, \langle n - \ell - 1, n - \ell - 1, 0 \rangle \right\} \\ & \cup \left\{ \langle -\frac{n}{2} + 1, 0, 1, \pm \frac{n}{2} \rangle, \langle -1, 0, \frac{n}{2} - 1, \pm \frac{n}{2} \rangle \right\} \end{aligned}$$

In particular note that for the first m difference cycles there is no difference between the tetrahedra containing 0 in M and the ones in M_k respectively.

Since M and M_k all have a transitive automorphism group, all vertex links within each individual complex are isomorphic and hence it suffices to look at the link of vertex 0 in order to verify that M or M_k is a combinatorial manifold. Since $(1 : \frac{n}{2} - 1 : 1 : \frac{n}{2} - 1)$ is part of M , we know that the link of vertex 0 appears as indicated in Figure 3.1 on the top left hand side, where the rest of the link fills the grey area, and all vertices v in the interior of the grey area are labelled by $v - n$ whenever $v > \frac{n}{2}$ (note that $(1 : \frac{n}{2} - 1 : 1 : \frac{n}{2} - 1)$ is a short orbit of length $\frac{n}{2}$). Now, if we look at the vertex link of M_k , $k > 0$, the fact that $d_i^0 + d_i^1 + d_i^2 \leq \frac{n}{2}$ for all difference cycles d_i together with the labelling convention assures that all vertex labels in the interior of the square surrounding the grey area remain unchanged. Outside the grey area the link grows by $2k$ triangles. By considering that the number of vertices of M_k is $n + k$ it is easy to verify by looking at Figure 3.1 on the top right (the vertex link of M_1) and on the bottom (the vertex link of M_k) that the vertex link of M_k is again a sphere for all $k > 0$.

Now assume that for at least one of the difference cycles d_i of M we have $d_i^0 + d_i^1 + d_i^2 > \frac{n}{2}$. If $(1 : \frac{n}{2} - 1 : 1 : \frac{n}{2} - 1)$ is part of M we can write the link of vertex 0 of M as before (see Figure 3.1 top left). Now look at the triangle $\langle d_i^0, d_i^0 + d_i^1, d_i^0 + d_i^1 + d_i^2 \rangle$. By construction (cf. the first part of the proof), the vertex $d_i^0 + d_i^1 + d_i^2$ is written as $-n + d_i^0 + d_i^1 + d_i^2$ and lies in the interior of the grey area. On the other hand we have $d_i^0 + d_i^1 + d_i^2 = \frac{n}{2} + k_0$ for some integer $k_0 \geq 1$ which lies on the boundary ($k_0 = 1$, see Figure 3.1 top right) or on the outside ($k_0 > 1$, see Figure 3.1 on the bottom) of the grey area. Hence the vertex $-n + d_i^0 + d_i^1 + d_i^2 = \frac{n}{2} + k_0$ is singular in the vertex link of 0 in M_{k_0} and M_{k_0} cannot be a combinatorial manifold.

By the same arguments as presented above, the vertex link of a manifold of the form M_k with $n + k$ vertices must look like the vertex link on the bottom of Figure 3.1 which thus can be reduced to a manifold of the form M_0 with n vertices.

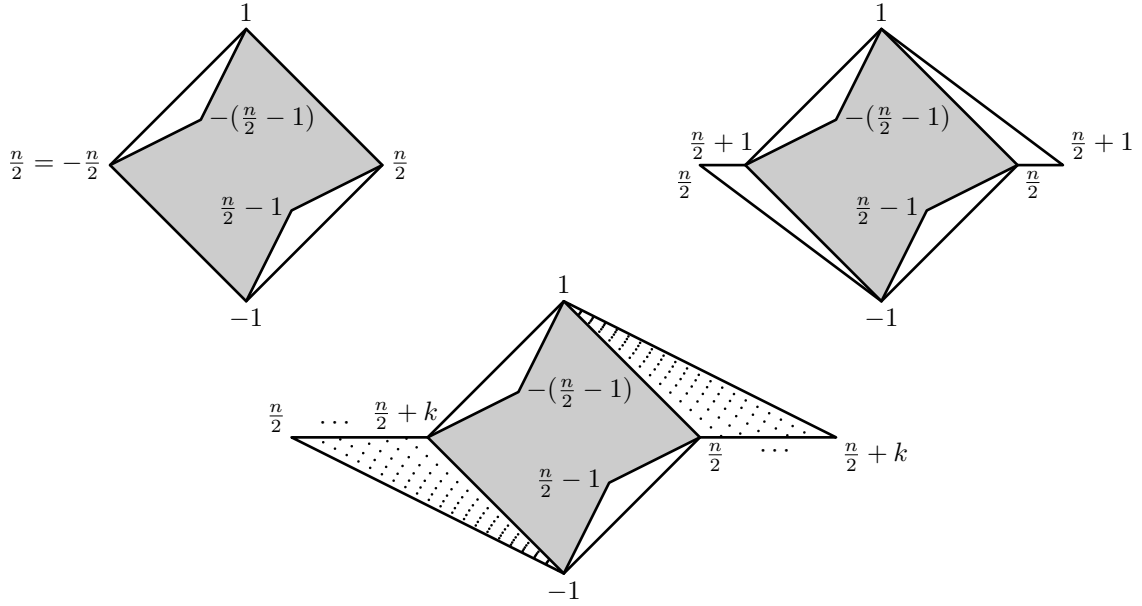


Figure 3.1: Link of vertex 0 of M (top left), M_1 (top right) and M_k (bottom).

Furthermore, the link of vertex 0 of M_k contains all vertices $\frac{n}{2}, \frac{n}{2}+1, \dots, n+k$. On the other hand, it contains all vertices $-\frac{n}{2}, -\frac{n}{2}+1, \dots, -1, 1, \dots, \frac{n}{2}$ if and only if M is neighbourly. Hence M_k is neighbourly if and only if M is neighbourly. \square

Remark 1. It seems that infinite series of combinatorial 3-manifolds as described in Theorem 1.1 usually contain one further combinatorial 3-manifold with $n-1$ vertices given by

$$M_{-1} = \{d_{1,-1}, \dots, d_{m,-1}\}$$

where $d_{i,-1} := (d_i^0 : d_i^1 : d_i^2 : d_i^3 - 1)$. In general, these manifolds then no longer share common difference cycles with the cyclic polytopes. However, in many cases the manifolds M_{-1} fit into the rest of the family in terms of the topological type.

The question of whether or not such a member M_{-1} always occurs or if families can be constructed where M_{-1} is not a combinatorial manifold is interesting but has to be left open at this point.

4 A 3-parameter family of combinatorial 3-manifolds

The aim of this section is to proof Theorems 1.3 and 1.4, and to prepare the proof of Theorem 1.2.

Theorem 1.1 allows us to find large numbers of infinite series of neighbourly combinatorial 3-manifolds. However, a priori it is not clear which of the families obtained by Theorem 1.1 actually describe an infinite family of distinct manifolds. Indeed, existing infinite series of combinatorial 3-manifolds suggest that most such families consist of infinitely many triangulations of only very few distinct topological 3-manifolds (cf. [25, Section 4.5.1] or [16]). Thus to obtain infinite families of interesting 3-manifolds requires more work.

The 3-parameter family of cyclic combinatorial 3-manifolds given in Theorem 1.2 was constructed by hand, by extending and combining various one-parameter families of interesting combinatorial 3-manifolds found by applying Theorem 1.1 and the census of cyclic combinatorial

3-manifolds from [26]. The subsequent analysis of the complexes was assisted by computer, using the computational topology software `simpcomp` [8, 9, 10] and the combinatorial recognition routines in `Regina` [6, 5].

4.1 Construction of the family

In what follows, we construct a 3-parameter family $M(p, q, r)$ of combinatorial 3-manifolds with transitive cyclic automorphism group, p and q co-prime positive integers, and r a non-negative integer. $M(p, q, r)$ consists of a base triangulation $B(p, q, r)$ and, for $r > 0$, three collections of solid tori $F_1(p, q, r)$, $F_2(p, q, r)$ and $F_3(p, q, r)$, each of which may consist of several solid tori, and each of which has compatible boundary with $B(p, q, r)$. These solid tori are then glued to $B(p, q, r)$ in order to give a closed combinatorial 3-manifold, hence

$$M(p, q, r) = B(p, q, r) \cup F_1(p, q, r) \cup F_2(p, q, r) \cup F_3(p, q, r).$$

We will see that, for $r > 0$, $B(p, q, r)$ is homeomorphic to a bundle over a punctured surface such that the solid tori $F_i(p, q, r)$, $1 \leq i \leq 3$, provide exceptional fibres.

For $r = 0$, $F_3(p, q, 0)$ is not a solid torus but a collection of pq tetrahedra glued together along common edges. Nonetheless, $M(p, q, 0)$ is still a combinatorial manifold.

Recall that we identify the vertices of $M(p, q, r)$ with the elements of \mathbb{Z}_{2pq+r} and all calculations involving the vertex labels are modulo $2pq+r$. In particular, a vertex denoted by $-v$, $pq \leq v \leq 2pq+r$, is interpreted as vertex $2pq+r-v$ in the naming convention explained in the proof of Theorem 1.1.

To construct $B(p, q, r)$, note that p and q are co-prime and hence there exist integers $m \in \{1, 2, \dots, q-1\}$ and $k \in \{1, 2, \dots, p-1\}$ such that $mp - kq = 1$. The base $B(p, q, r)$ is then given by

$$B(p, q, r) = \{(1 : kq : (q-m)p : pq+r), (1 : kq : pq+r : (q-m)p), (1 : pq+r : kq : (q-m)p)\}.$$

To construct the first collection of solid tori $F_1(p, q, r)$ let us assume w.l.o.g. that $(p-k)q > kq$ (if $kq \geq (p-k)q$ the initial arguments of the Euclidean algorithm below are interchanged resulting in a similar construction).

If the Euclidean algorithm is run with input kq and $(p-k)q$ this yields a series of equations

$$\begin{array}{lll} a_1 = (p-k)q; & b_1 = kq; & \\ a_2 = a_1 - b_1; & b_2 = b_1; & \\ \dots & \dots & \\ \dots & \dots & \\ \text{if } a_i > b_i : & a_{i+1} = a_i - b_i; & b_{i+1} = b_i; \\ \text{if } a_i < b_i : & a_{i+1} = b_i - a_i; & b_{i+1} = a_i; \\ \dots & \dots & \\ \dots & \dots & \\ a_{N((p-k)q, kq)} = q; & b_{N((p-k)q, kq)} = q; & \end{array} \quad (4.1)$$

(note that by construction, the greatest common divisor of kq and $(p-k)q$ is q). Then F_1 is given by

$$F_1(p, q, r) = \{(b_i : a_i : b_i : 2pq - 2b_i - a_i + r) \mid 1 \leq i \leq N((p-k)q, kq)\}.$$

The construction of $F_2(p, q, r)$ is analogous. Let w.l.o.g. $(q-m)p > mp$. The greatest common divisor of $(q-m)p$ and mp is p and if (c_i, d_i) , $1 \leq i \leq N((q-m)p, mp)$, is the sequence of integer pairs from the Euclidean algorithm as described above then F_2 is given by

$$F_2(p, q, r) = \{(d_i : c_i : d_i : 2pq - 2d_i - c_i + r) \mid 1 \leq i \leq N((q-m)p, mp)\}.$$

Finally, the complex $F_3(p, q, r)$ is a subset of the boundary complex of the cyclic 4-polytope, namely

$$F_3(p, q, r) = \{(1 : pq - 1 + i : 1 : pq - 1 + r - i) \mid 0 \leq i \leq \lfloor r/2 \rfloor + 1\},$$

it is a solid torus for $r > 0$ and consists of the single short difference cycle $(1 : pq - 1 : 1 : pq - 1)$ for $r = 0$.

Lemma 4.1. *For every pair of co-prime p and q , $2 \leq p < q$, and $r \geq 0$, the simplicial complex $M(p, q, r)$ is a combinatorial 3-manifold.*

Proof. See Figures 4.1 and 4.2 for drawings of the vertex link of vertex 0 of $M(p, q, r)$ - a combinatorial 2-sphere. By the transitive symmetry we know that all vertex links are combinatorially isomorphic to the link of vertex 0 and hence all vertex links of $M(p, q, r)$ are homeomorphic to the 2-sphere. \square

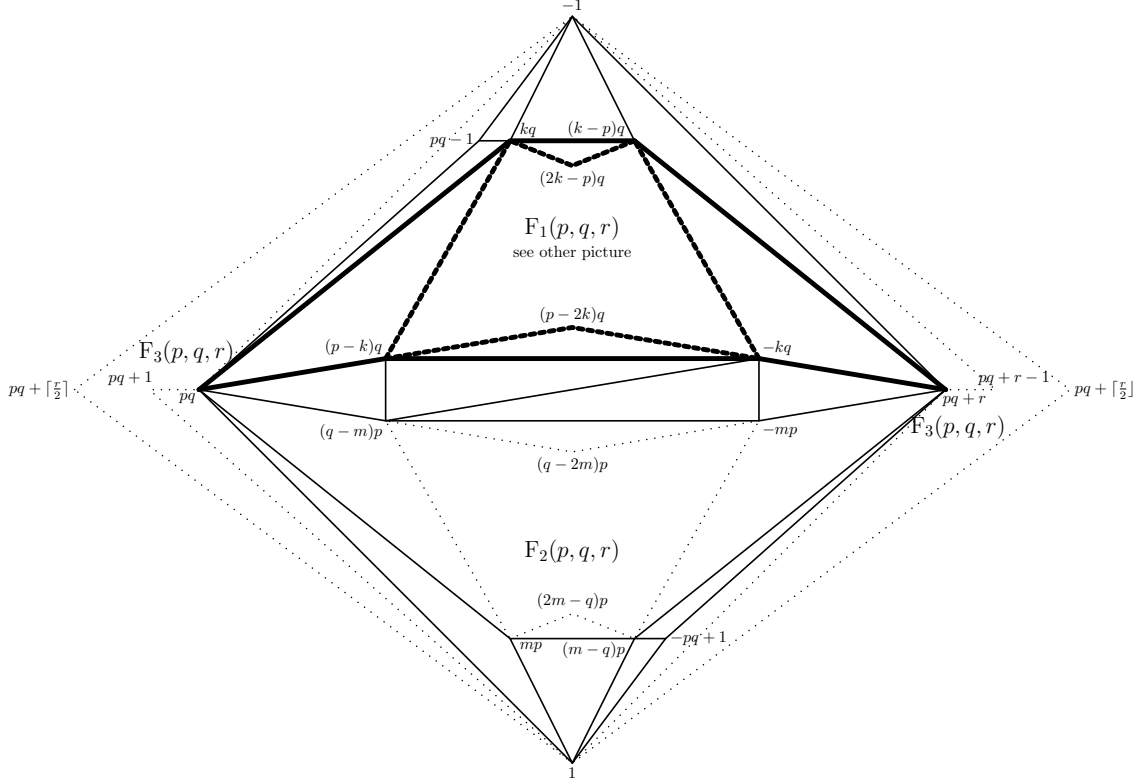


Figure 4.1: Vertex link of vertex 0 of $M(p, q, r)$. The solid lines represent the triangles belonging to $B(p, q, r)$, the dotted lines represent the triangles belonging to the first difference cycle of $F_i(p, q, r)$, $1 \leq i \leq 3$, as indicated. See Figure 4.2 for a more detailed drawing of the region $F_1(p, q, r)$.

In Section 5 we prove that the combinatorial 3-manifolds $M(p, q, r)$, $r > 0$, are in fact combinatorial Seifert fibred spaces with changing topological types and, for $r = 0$, homeomorphic to $(\mathbf{S}^2 \times \mathbf{S}^1)^{\#(p-1)(q-1)}$. However, let us first determine some other interesting attributes of these combinatorial manifolds.

4.2 An upper bound for the Heegaard genus of $M(p, q, r)$

In this section we determine an upper bound for the Heegaard genus of $M(p, q, r)$ using rsl -functions (cf. Section 2.5 and [15]).

Theorem 4.2. *For all $M(p, q, r)$, p and q co-prime, the rsl -function*

$$f : M(p, q, r) \rightarrow [0, 1]; \quad v \mapsto \frac{v}{2pq + r - 1}$$

has exactly $2(p-1)(q-1) + 2$ critical points.

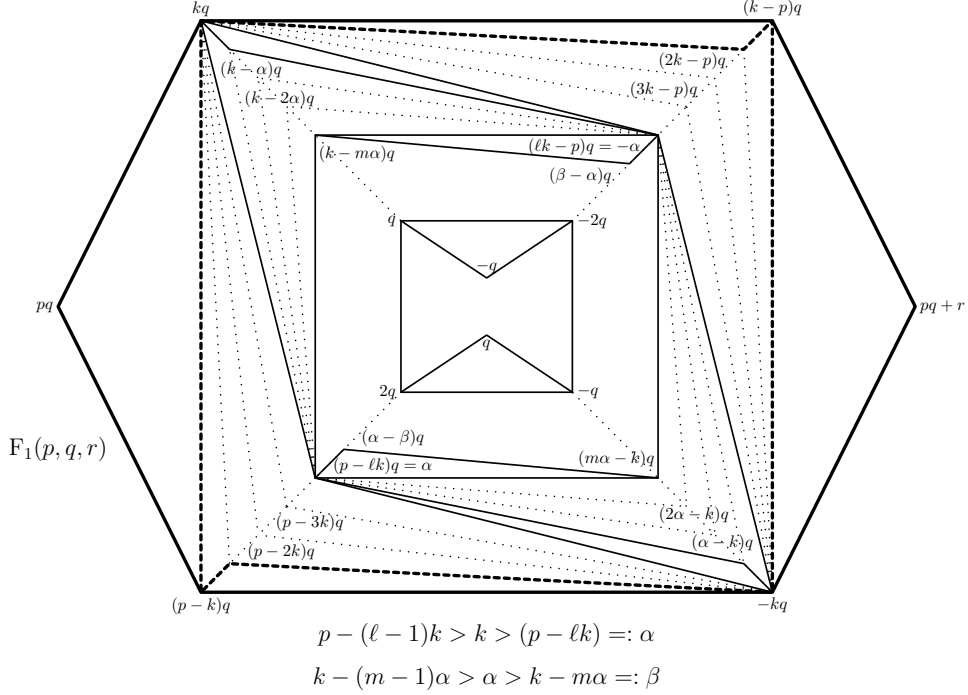


Figure 4.2: The disc $F_1(p, q, r)$ from Figure 4.1 in more detail.

In order to prove Theorem 4.2 we first establish some observations about critical points of index 1 of f . In doing so we sometimes abuse notation and refer to a non-critical point as a critical point of index i and multiplicity 0. Moreover, the set of faces of a simplicial complex C whose vertices are entirely contained in a subset $\{v_1, \dots, v_k\}$ of the vertices of C are denoted by $\text{span}_{\{v_1, \dots, v_k\}}(C)$. Finally, in all of the following calculations which require the choice of a field we use the field with two elements \mathbb{F}_2 .

Lemma 4.3. *Vertex v of $M(p, q, r)$, $0 \leq v \leq 2pq + r - 1$, is critical of index 1 and multiplicity*

$$\tilde{\beta}_0(\text{span}_{\{-v, -v+1, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0)))$$

with respect to f , where $\tilde{\beta}_0 = \beta_0 - 1$ is the reduced Betti number of index 0 denoting the number of connected components minus 1.

Proof. The multiplicity of a critical point v of index i with respect to an rsl-function $g : M(p, q, r) \rightarrow [0, 1]$ is given by the dimension of the i -th relative homology $\dim_{\mathbb{F}_2} H_i(M_v, M_v \setminus \{v\}, \mathbb{F}_2)$ where $M_v := \{x \in M(p, q, r) \mid g(x) \leq g(v)\}$.

This is equivalent to looking at the $(i-1)$ -th reduced Betti number $\tilde{\beta}_{i-1}$ of $\text{span}_{V_v}(\text{lk}_{M(p, q, r)}(v))$ where V_v is the subset of vertices w such that $g(w) < g(v)$. For the rsl-function $f(v) = \frac{v}{2pq+r-1}$, $v \in \{0, 1, \dots, 2pq + r - 1\}$, this means that vertex v is critical of index 1 with multiplicity $\tilde{\beta}_0(\text{span}_{\{0, 1, \dots, v-1\}}(\text{lk}_{M(p, q, r)}(v)))$, and since $M(p, q, r)$ has a vertex transitive cyclic automorphism group we have $\text{span}_{\{0, 1, \dots, v-1\}}(\text{lk}_{M(p, q, r)}(v)) \cong \text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$ which proves the result. \square

Lemma 4.4. *If vertex $-v$ of $M(p, q, r)$ is not contained in $\text{lk}_{M(p, q, r)}(0)$, then vertex v is critical of the same index with the same multiplicity as vertex $v - 1$ with respect to f .*

Proof. If $-v \notin \text{lk}_{M(p, q, r)}(0)$, then

$$\text{span}_{\{-v+1, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0)) = \text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$$

and hence

$$\text{span}_{\{1, \dots, v-2\}}(\text{lk}_{M(p,q,r)}(v-1)) = \text{span}_{\{1, \dots, v-1\}}(\text{lk}_{M(p,q,r)}(v)).$$

□

Lemma 4.5. *The complex $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{F_i(p,q,r)}(0))$, $1 \leq i \leq 2$, is connected for all integers $-pq \leq -v \leq -1$.*

Proof. We prove Lemma 4.5 for $F_1(p, q, r)$. The proof that $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{F_2(p,q,r)}(0))$ is connected for $-pq \leq -v \leq -1$ is completely analogous.

Recall that

$$F_1(p, q, r) = \{(b_i : a_i : b_i : 2pq - 2b_i - a_i + r) \mid 1 \leq i \leq N((p-k)q, kq)\},$$

where the a_i and b_i are given by the Euclidean algorithm.

Due to the symmetry in the difference cycles of F_1 , $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{F_1(p,q,r)}(0))$ is connected if and only if $\text{span}_{\{1, \dots, v\}}(\text{lk}_{F_1(p,q,r)}(0))$ is connected. Hence we focus on the latter and $1 \leq v \leq pq$.

All vertices of $\text{span}_{\{1, \dots, pq\}}(\text{lk}_{F_1(p,q,r)}(0))$ are of the form b_i , a_i , $a_i + b_i$ or $a_i + 2b_i$ and the edges are of the form $\langle b_i, a_i + b_i \rangle$, $\langle a_i, a_i + b_i \rangle$ or $\langle a_i + b_i, a_i + 2b_i \rangle$ for some i , $1 \leq i \leq N((p-k)q, kq)$.

We have $a_i + b_i = \max\{a_{i-1}, b_{i-1}\}$ (this follows from one step of the Euclidean algorithm given by Equation (4.1)), and $a_i + 2b_i = \max\{a_{i-2}, b_{i-2}\}$ which can be seen by considering the following four cases.

- **Case $a_{i-2} - b_{i-2} > b_{i-2}$:** We have $a_i = a_{i-2} - 2b_{i-2}$ and $b_i = b_{i-2}$ and the statement follows.
- **Case $a_{i-2} > b_{i-2}$ and $a_{i-2} - b_{i-2} < b_{i-2}$:** This results in $a_{i-1} = a_{i-2} - b_{i-2}$ and $b_{i-1} = b_{i-2}$ followed by swapping the variables yielding $a_i = b_{i-1} - a_{i-1} = 2b_{i-2} - a_{i-2}$ and $b_i = a_{i-2} - b_{i-2}$, and hence $a_i + 2b_i = a_{i-2}$.
- **Case $a_{i-2} < b_{i-2}$ and $b_{i-2} - a_{i-2} > a_{i-2}$:** Here we first swap variables, thus, $a_{i-1} = b_{i-2} - a_{i-2}$ and $b_{i-1} = a_{i-2}$ followed by $a_i = b_{i-2} - 2a_{i-2}$ and $b_i = a_{i-2}$ and all together $a_i + 2b_i = b_{i-2}$.
- **Case $a_{i-2} < b_{i-2}$ and $b_{i-2} - a_{i-2} < a_{i-2}$:** Now we have to swap variables twice resulting in $a_{i-1} = b_{i-2} - a_{i-2}$, $b_{i-1} = a_{i-2}$ and $a_i = 2a_{i-2} - b_{i-2}$, $b_i = b_{i-2} - a_{i-2}$ and hence $a_i + 2b_i = b_{i-2}$.

Finally for $i \leq 2$ it follows that $a_1 + 2b_1 = pq$, and $a_2 + 2b_2 = a_1 + b_1 = \max\{(p-k)q, kq\}$. As a result we have

$$\{a_1, b_1, a_2, b_2, \dots, a_{N((p-k)q, kq)}, b_{N((p-k)q, kq)}\} = \{\max\{a_1, b_1\}, \dots, \max\{a_{N((p-k)q, kq)}, b_{N((p-k)q, kq)}\}\},$$

where $\max\{a_i, b_i\} > \max\{a_{i+1}, b_{i+1}\}$.

It follows that $\text{lk}_{F_1(p,q,r)}(0)$ contains edges of the form $\langle \max\{a_{i-1}, b_{i-1}\}, \max\{a_{i-2}, b_{i-2}\} \rangle$ for all $1 \leq i \leq N((p-k)q, kq)$. Hence there exist a path meeting all vertices of $\text{span}_{\{1, \dots, pq\}}(\text{lk}_{F_1(p,q,r)}(0))$ in increasing / decreasing order. By symmetry this also holds for vertices $-pq \leq v \leq -1$, and $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{F_1(p,q,r)}(0))$ is connected for all $-pq \leq -v \leq -1$. □

Lemma 4.6. *The complex $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p,q,r)}(0))$ is connected for all $-2pq - r + 1 \leq -v \leq -pq$.*

Proof. By looking at Figure 4.1 we can see that $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p,q,r)}(0))$ is connected for all $-pq - r \leq -v \leq -pq$ and $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{B(p,q,r)}(0))$ is connected for all $-2pq - r + 1 \leq -v \leq -pq$. Moreover, from the proof of Lemma 4.5 we can see that both $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{F_i(p,q,r)}(0))$, $1 \leq i \leq 2$, are connected for $-2pq - r + 1 \leq -v \leq -pq$ and attached to $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{B(p,q,r)}(0))$. All together it follows that $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p,q,r)}(0))$ is connected for $-2pq - r + 1 \leq -v \leq -pq$. □

Proof of Theorem 4.2. Since $M(p, q, r)$ contains all edges $\langle v, v+1 \rangle$, $0 \leq v \leq 2pq+r-2$, and $M(p, q, r)$ is a combinatorial manifold, f has exactly one critical vertex of index 0 and exactly one critical vertex of index 3.

Now, by Lemma 4.3, the critical vertices of index 1 of f and their multiplicities can be determined by counting the number of connected components minus 1 of

$$\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$$

for all $1 \leq v \leq 2pq + r - 1$. By Lemma 4.4 and Lemma 4.6, $\text{span}_{\{-v, -v+1, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$ is connected for $v \geq pq$ and so no vertex $v \geq pq$ can be a critical vertex of f of index ≥ 1 .

Furthermore, by Lemma 4.4 and Lemma 4.5 for each vertex $p \leq v < pq$ the complex $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$ has at most 3 connected components, and hence v is critical of index 1 with multiplicity at most 2.

Case 1: Let $(q - m)p > mp$. Recall that $mp = kq + 1$ and thus $(p - k)q > kq$. It follows that for all vertices $p \leq v \leq q - 1$ the complex $\text{span}_{\{-v, -v+1, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$ has two connected components (cf. Figure 4.1) and hence these vertices are critical of index 1 and multiplicity 1 (cf. Lemma 4.4), for vertices $q \leq v \leq kq$ we can see that $\text{span}_{\{-v, -v+1, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$ has three connected components and thus we have critical vertices of index 1 and multiplicity 2. For $mp \leq v \leq (q - m)p$ we again have two connected components and thus critical vertices of index 1 and multiplicity 1 and for all other vertices the complex is connected (cf. Lemma 4.6). All together there are $(p - 1)(q - 1)$ critical points of index 1.

Case 2: Let $(q - m)p \leq mp$. The same argument as before shows that vertices $p \leq v \leq q - 1$ are critical of index 1 and multiplicity 1, vertices $q \leq v \leq (q - m)p$ are critical of index 1 and multiplicity 2 and vertices $(p - k)q \leq v \leq kq$ are critical of index 1 and multiplicity 1, which also results in $(p - 1)(q - 1)$ critical points of index 1.

Now, since the alternating sum over all critical points counted by multiplicity equals the Euler characteristic (which has to be 0), and f has only one critical point of index 0 and 3 each, the number of critical points of index 1 must equal the number of critical points of index 2. All together f has $(p - 1)(q - 1)$ critical points of index 1 and 2 each and thus $2(p - 1)(q - 1) + 2$ critical points in total which proves the result. \square

Theorem 1.3 now follows as a simple corollary of Theorem 4.2.

4.3 The homology groups of $M(p, q, r)$

By the proof of Theorem 4.2 the rsl-function

$$f : M(p, q, r) \rightarrow [0, 1]; \quad v \mapsto \frac{v}{2pq + r - 1}$$

has $(p - 1)(q - 1)$ critical points of index 1. Furthermore, Lemma 4.6 together with the transitive cyclic symmetry of $M(p, q, r)$ tells us that only vertices $1 < v < pq$ can be critical of index 1 and again by the transitive cyclic symmetry it follows that all these critical points of index 1 have to pair with critical vertices $v \geq pq$. All together it follows that $B_- = \text{span}_{0, 1, \dots, pq-1}(M(p, q, r))$ and $B_+ = \text{span}_{pq, pq+1, \dots, 2pq+r-1}(M(p, q, r))$ must be “handlebodies”¹ of genus $(p - 1)(q - 1)$.

Thus the topological type of $M(p, q, r)$ is determined by how a set of $(p - 1)(q - 1)$ simple closed curves in B_- forming a basis of the first homology group is glued to B_+ .

A basis of the first homology group of B_- can be found by the observations made in the previous section (in particular, cf. Lemma 4.3 and the proof of Theorem 4.2): for all v , $p \leq v < pq$, we connect distinct connected components of $\text{span}_{\{-v, \dots, -1\}}(\text{lk}_{M(p, q, r)}(0))$ (whenever they exist) by a path in $\text{span}_{\{-v, \dots, -1\}}(M(p, q, r))$.

¹ B_- and B_+ might contain isolated edges and triangles and are thus only homotopic to a handlebody. However, there is always a small neighbourhood of B_- and B_+ which is a proper handlebody.

One possible choice for such a basis of $H_1(B_-)$ is

$$\langle v, v-1, v-2, \dots, v-p, v \rangle$$

for $p \leq v \leq kq$ and

$$\langle v, v-1, v-2, \dots, v-q, v \rangle$$

for $q \leq v \leq (q-m)p$.

Lemma 4.7. *Let $[c]$ be an element of $H_1(M(p, q, r))$. Then*

$$c \simeq c + pq,$$

where for any path $c = \langle v_1, v_2, \dots, v_r \rangle$ and $x \in \mathbb{Z}_n$, the sum $c + x$ denotes the path obtained from c by adding $x \bmod n$ component-wise, that is, $c + x = \langle (v_1 + x) \bmod n, (v_2 + x) \bmod n, \dots, (v_r + x) \bmod n \rangle$.

Proof. We show that $c \simeq c + pq$ for all basis elements in $H_1(B_-)$ and hence for a generating system of $H_1(M(p, q, r))$.

This is done by using triangles contained in the difference cycles of $F_1(p, q, r)$ and $B(p, q, r)$ to gradually transform $\langle v, v-1, v-2, \dots, v-p, v \rangle$ into $\langle pq+v, pq+v-1, pq+v-2, \dots, pq+v-p, pq+v \rangle$. The proof for generating elements of the form $\langle v, v-1, v-2, \dots, v-q, v \rangle$ is analogous.

Let $m = N((q-m)p, mp)$ and $c_i := \max\{a_{m-i}, b_{m-i}\}$. Then

$$\begin{aligned} \langle v, v-1, \dots, v-p, v \rangle &\simeq \langle v, pq+v, v-1, pq+v-1, v-2, pq+v-2, \dots, pq+v-p+1, v-p, v \rangle \\ &\simeq \langle v, pq+v, v-1, pq+v-1, v-2, pq+v-2, \dots, pq+v-p+1, pq+v-p, v-p, v \rangle \\ &\simeq \langle v, pq+v, pq+v-1, pq+v-2, \dots, pq+v-p+1, pq+v-p, v-p, v \rangle \\ &\simeq \langle v, pq+v, pq+v-1, pq+v-2, \dots, pq+v-p, v-c_1, v+c_1, v \rangle \\ &\simeq \langle v, pq+v, \dots, pq+v-p, v-c_1, v+c_1, v+c_2, \dots, v+c_m, v \rangle \\ &\simeq \langle v, pq+v, \dots, pq+v-p, v-c_1, c_1+v, \dots, c_m+v, \max\{(q-m)p, mp\}+v, pq+v, v \rangle \\ &\simeq \langle v, pq+v, \dots, pq+v-p, v-c_1, pq+v-p, pq+v, v \rangle \\ &\simeq \langle pq+v, \dots, pq+v-p, v-c_1, pq+v-p, pq+v \rangle \\ &\simeq \langle pq+v, \dots, pq+v-p, pq+v \rangle. \end{aligned} \quad \square$$

Together with the cyclic symmetry, the above observation allows us to analyse $M(p, q, r)$ in further detail. In particular, it follows from Lemma 4.7 that given p and q , the homology of $M(p, q, r)$ only depends on $r \bmod pq$.

As a special case, if $r \equiv 0 \bmod pq$ we can deduce that $H_1(M(p, q, r)) = \mathbb{Z}^{(p-1)(q-1)}$, and if $\gcd(p, r) = \gcd(q, r) = 1$ then all generators of $H_1(B_-)$ are identified in $H_1(M(p, q, r))$ eventually resulting in trivial homology. More generally, if $a = \gcd(p, r)$ and $b = \gcd(q, r)$ then

$$H_1(M(p, q, r)) = \mathbb{Z}^{(a-1)(b-1)} \oplus \mathbb{Z}_{p/a}^{b-1} \oplus \mathbb{Z}_{q/b}^{a-1} \quad (4.2)$$

and since all $M(p, q, r)$ are orientable we have

$$H_*(M(p, q, r)) = (\mathbb{Z}, \mathbb{Z}^{(a-1)(b-1)} \oplus \mathbb{Z}_{p/a}^{b-1} \oplus \mathbb{Z}_{q/b}^{a-1}, \mathbb{Z}^{(a-1)(b-1)}, \mathbb{Z}).$$

We do not prove the claims made above since they independently follow from the topological types of $M(p, q, r)$ shown in Section 5. However, the specific structure of $M(p, q, r)$ given by Theorem 4.2 and Lemma 4.7 gives rise to an interesting and rarely observed connection between the automorphism group of $M(p, q, r)$ in the case $p = 2$, q prime, $r \equiv 0 \bmod pq$, and its first homology group which is discussed in the following section.

4.4 Action of the automorphism group on the homology of $M(2, q, 2kq)$

In this section we present a number of non-trivial group representations of the cyclic group

$$\text{Aut}(M(2, q, 2kq)) = \langle g \rangle$$

with $g = (0, 1, \dots, 2q(k+2))$, q prime, into the free \mathbb{Z} -module

$$H_1(M(2, q, 2kq)) = \mathbb{Z}^{q-1},$$

$k \geq 0$. In particular, we give a proof of Theorem 1.4.

This is done by applying Lemma 4.7 to a suitable choice of a basis of $H_1(M(2, q, 2kq))$ and following the construction of finite order integer matrices as described in [14].

Proof of Theorem 1.4. Note that by Lemma 4.7 for every cycle c in $M(2, q, 2kq)$ we have $c \simeq c + 2q$. Hence the size of the image of every action

$$\rho : \text{Aut}(M(2, q, 2kq)) \rightarrow \text{SL}(q-1, \mathbb{Z})$$

divides $2q$ and in particular

$$|\rho(\text{Aut}(M(2, q, 2kq)))| \leq 2q.$$

In particular, $\rho(g)$ is an integer matrix of order $\leq 2q$.

Following the observations made in the last section, a basis of $H_1(M(2, q, 2kq))$ is given by the cycles

$$a_{v-1} = \langle v, v-1, v-2, v \rangle$$

for $2 \leq v \leq q$. Thus by construction we have

$$g \cdot a_i = a_{i+1},$$

$2 \leq i < q$, where g acts on the cycles of $M(2, q, 2kq)$ by adding 1 modulo $2q(k+2)$ to each entry of the cycle.

Moreover, up to similarity transformations, the only matrix $M \in \text{SL}(q-1, \mathbb{Z})$ of finite order $2q$ is of the form

$$M = \begin{pmatrix} 0 & \dots & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

See [14] where M is described in more detail. As a side note, in [4] a similar finite-order integer matrix (of order q) occurs in a construction of d -dimensional combinatorial tori.

Note that the first $q-2$ columns of M are compatible with the above choice for a basis of $H_1(M(2, q, 2kq))$ and, in order to prove Theorem 1.4, it remains to show that

$$g \cdot a_{q-1} = a_2^{-1} a_3 a_4^{-1} a_5 \dots a_{q-2}^{-1} a_{q-1}.$$

We have

$$\begin{aligned} M(2, q, 2kq) = \{ & (1 : q : q-1 : 2q(k+1)), \\ & (1 : q : 2q(k+1) : q-1), \\ & (1 : 2q(k+1) : q : q-1), \\ & (1 : 2q-1 : q-1 : 2q(k+1)-1)), \\ & (q-1 : 2 : q-1 : 2q(k+1)), \\ & (2 : q-3 : 2 : 2q(k+2)-q-1)), \\ & (2 : q-5 : 2 : 2q(k+2)-q+1)), \\ & (2 : q-7 : 2 : 2q(k+2)-q+3)), \\ & \dots, \\ & (2 : 2 : 2 : 2q(k+2)-6)) \}. \end{aligned}$$

In particular, we have the following triangle relations:

$$\begin{aligned}
(1 : q) &\leftrightarrow (q + 1), \\
(q - 1 : 1) &\leftrightarrow (q), \\
(q : q - 1) &\leftrightarrow (2q - 1), \\
(1 : 2q - 1) &\leftrightarrow (2q), \\
(q - 1 : 2) &\leftrightarrow (q + 1), \\
(2 : q - 1) &\leftrightarrow (q + 1), \\
(q - 3 : 2) &\leftrightarrow (q - 1), \\
(2 : q - 3) &\leftrightarrow (q - 1), \\
\vdots &\vdots, \\
(2 : 2) &\leftrightarrow (4).
\end{aligned}$$

For the basis elements of $H_1(M(2, q, 2kq))$ this translates to

$$\begin{aligned}
a_{v-1}^{-1} &\simeq \langle v, v - 2, v - 1, v \rangle \\
&\simeq \langle v, 0, v - 2, v - 1, v \rangle \\
&\simeq \langle 0, v - 2, v - 1, v, 0 \rangle
\end{aligned}$$

and

$$\begin{aligned}
a_v &\simeq \langle v + 1, v, v - 1, v + 1 \rangle \\
&\simeq \langle v + 1, v + q + 1, v, v - 1, v + 1 \rangle \\
&\simeq \langle v + 1, v + q + 1, v + 2, v, v - 1, v + 1 \rangle \\
&\simeq \langle v + 1, v + q + 1, v + 2, 0, v, v - 1, v + 1 \rangle \\
&\simeq \langle 0, v, v - 1, v + 1, v + q + 1, v + 2, 0 \rangle
\end{aligned}$$

for $v \in \{2, 4, \dots, q - 1\}$, and thus

$$\begin{aligned}
a_{v-1}^{-1} a_v &\simeq \langle 0, v - 2, v - 1, v, 0 \rangle \langle 0, v, v - 1, v + 1, v + q + 1, v + 2, 0 \rangle \\
&\simeq \langle 0, v - 2, v - 1, v + 1, v + q + 1, v + 2, 0 \rangle \\
&\simeq \langle 0, v - 2, v - 1, v + 1, v, v + q + 1, v + 2, 0 \rangle \\
&\simeq \langle 0, v - 2, v - 1, v + 1, v, v + 2, 0 \rangle \\
&\simeq \langle 0, v - 2, v - 1, v + 1, v, 0 \rangle.
\end{aligned}$$

Putting these pieces together this results in

$$\begin{aligned}
(a_1^{-1} a_2) \dots (a_{q-2}^{-1} a_{q-1}) &\simeq \langle 0, 0, 1, 3, 2, 0 \rangle \langle 0, 2, 3, 5, 4, 0 \rangle \dots \langle 0, v - 2, v - 1, v + 1, v, 0 \rangle \\
&\simeq \langle 0, 1, 3, 5, \dots, q - 2, q, q - 1, 0 \rangle \\
&\simeq \langle 0, 1, q, q - 1, 0 \rangle \\
&\simeq \langle 0, 1, q + 1, q, q - 1, 0 \rangle \\
&\simeq \langle 0, q + 1, q, q - 1, q + 1, 0 \rangle \\
&\simeq \langle q + 1, q, q - 1, q + 1 \rangle \\
&\simeq g \cdot a_{q-1}.
\end{aligned}$$

□

5 The topological types of $M(p, q, r)$

In this section we prove Theorem 1.2. That is, we show that $M(p, q, r)$ is homeomorphic to the Seifert fibred spaces of type

$$\text{SFS}[(\mathbb{T}^2)^{\#(a-1)(b-1)/2} : (-p/a, b_1)^b (q/b, b_2)^a (-r/(ab), b_3)]$$

with $a = \gcd(p, r)$ and $b = \gcd(q, r)$, $r > 0$.

In particular, we show that $M(p, q, r)$ is homeomorphic to the Brieskorn homology sphere $\Sigma(p, q, r)$ whenever p, q and r are co-prime, $M(2, q, 2)$ is homeomorphic to the lens space $L(q, 1)$, and that, in the limit case $r = 0$, we have $M(p, q, 0) \cong (\mathbf{S}^2 \times \mathbf{S}^1)^{\#(p-1)(q-1)}$.

The proof is given as a corollary of the following five observations.

1. The Seifert fibrations given in Theorem 1.2 are well-defined, i.e., all invariants of Seifert fibred spaces satisfying the conditions of the theorem for the same triple (p, q, r) , $r > 0$, are isomorphic (cf. Lemma 5.1).
2. $M(p, q, r)$ is a combinatorial manifold for all $p, q \in \mathbb{N}$, p and q co-prime, and for all non-negative integers $r \geq 0$ (cf. Lemma 4.1).
3. For $a = \gcd(p, r)$ and $b = \gcd(q, r)$,
 - $F_1(p, q, r)$ is a triangulation of b disjoint copies of a solid torus,
 - $F_2(p, q, r)$ is a triangulation of a disjoint copies of a solid torus,
 - for $r > 0$, $F_3(p, q, r)$ is a triangulation of a single solid torus, and,
 - for $r = 0$, a collection of pq tetrahedra glued together along edges forming a solid torus pinched along edges.

Furthermore, the boundary of the meridian disc of each torus can be explicitly described (cf. Lemma 5.2).

4. For $r > 0$, $B(p, q, r)$ united with a small neighbourhood of the boundaries of $F_i(p, q, r)$, $1 \leq i \leq 3$, is homeomorphic to the Cartesian product of a circle with the orientable surface of genus $\frac{1}{2}(a-1)(b-1)$ with $b+a+1$ discs removed, where each of the $b+a+1$ boundary components corresponds to one boundary torus of $F_i(p, q, r)$, $1 \leq i \leq 3$ (cf. Lemma 5.3). In addition, the boundary curves of the $b+a+1$ meridian discs of $F_i(p, q, r)$, $1 \leq i \leq 3$, in $B(p, q, r)$ can be determined to be of the desired type (cf. Lemma 5.5).
5. For $r = 0$, $B(p, q, 0) \cup F_3(p, q, 0)$ united with a small neighbourhood of the boundaries of $F_i(p, q, 0)$, $1 \leq i \leq 2$, minus a small neighbourhood of $F_3(p, q, 0)$, is homeomorphic to the Cartesian product of a circle with the orientable surface of genus $\frac{1}{2}(p-1)(q-1)$ with $p+q+1$ discs removed. The boundary curves of the $p+q$ meridian discs of $F_i(p, q, 0)$, $1 \leq i \leq 2$, and the meridian disc of a thickened version of $F_3(p, q, 0)$ in $B(p, q, 0)$ can be determined to be of the desired type.

We first give detailed proofs of these five observations before we summarise them in order to prove Theorem 1.2.

Lemma 5.1. *Given positive integers $p, q, r \in \mathbb{N}$, $2 \leq p < q$ co-prime, $r > 0$, $a = \gcd(p, r)$ and $b = \gcd(q, r)$, then all Seifert fibrations*

$$\text{SFS}[(\mathbb{T}^2)^{\#(a-1)(b-1)/2} : (-p/a, b_1)^b (q/b, b_2)^a (-r/ab, b_3)]$$

satisfying

$$\left(\frac{b_1}{p} - \frac{b_2}{q} + \frac{b_3}{r}\right) \frac{pqr}{ab} = 1$$

are isomorphic. In particular, their underlying manifolds are homeomorphic.

Proof. The isomorphism type of a Seifert fibred space with exceptional fibres (a_i, b_i) , $1 \leq i \leq r$, does not change by simultaneously adding a_j to b_j and subtracting a_k from b_k for any pair of indices $1 \leq j, k \leq r$, or by changing the sign of all the a_i , $q \leq i \leq r$ (cf. [24]).

Now let p, q, r be fixed and b_i, b'_i , $1 \leq i \leq 3$, such that

$$\left(\frac{b_1}{p} - \frac{b_2}{q} + \frac{b_3}{r}\right) \frac{pqr}{ab} = 1 = \frac{pqr}{ab} \left(\frac{b'_1}{p} - \frac{b'_2}{q} + \frac{b'_3}{r}\right).$$

In particular, this means that

$$\frac{qr}{ab}(b_1 - b'_1) - \frac{pr}{ab}(b_2 - b'_2) + \frac{pq}{ab}(b_3 - b'_3) = 0 \quad (5.1)$$

and thus

$$\frac{r}{ab}(q(b_1 - b'_1) - p(b_2 - b'_2)) = \frac{-pq}{ab}(b_3 - b'_3).$$

Now note that $\gcd(\frac{r}{ab}, \frac{-pq}{ab}) = 1$ by construction and hence there exist an $\alpha \in \mathbb{Z}$ such that

$$(q(b_1 - b'_1) - p(b_2 - b'_2)) = \alpha \frac{pq}{ab} \quad \text{and} \quad (b_3 - b'_3) = \alpha \frac{r}{ab}.$$

In particular $q(b_1 - b'_1) - p(b_2 - b'_2) \equiv 0 \pmod{\frac{pq}{ab}}$ holds, and since furthermore $\gcd(\frac{q}{b}, \frac{p}{a}) = 1$, we have both $q(b_1 - b'_1) - p(b_2 - b'_2) \equiv 0 \pmod{\frac{p}{a}}$ and $q(b_1 - b'_1) - p(b_2 - b'_2) \equiv 0 \pmod{\frac{q}{b}}$ by the Chinese remainder theorem.

It follows that $(b_1 - b'_1)$ is a multiple of $\frac{p}{a}$, $(b_2 - b'_2)$ is a multiple of $\frac{q}{b}$, $(b_3 - b'_3)$ is a multiple of $\frac{r}{ab}$, by Equation (5.1) additions and subtractions sum up to zero, and thus the Seifert fibred spaces corresponding to (b_1, b_2, b_3) and (b'_1, b'_2, b'_3) are isomorphic. \square

Lemma 5.2. *Given positive integers $p, q, r \in \mathbb{N}$, $2 \leq p < q$ co-prime, $r \geq 0$, $a = \gcd(p, r)$ and $b = \gcd(q, r)$, we have:*

- $F_1(p, q, r) \cong \{1, 2, \dots, b\} \times (\mathbf{B}^2 \times \mathbf{S}^1)$ where the boundaries of the meridian discs $m_1^{(i)}$, $0 \leq i \leq b-1$, are given by the paths

$$\partial m_1^{(i)} = \langle i, kq + i, 2kq + i, \dots, (p-1)kq + i, pkq + i, (k-1)pq + i, (k-2)pq + i, \dots, i \rangle;$$

- $F_2(p, q, r) \cong \{1, 2, \dots, a\} \times (\mathbf{B}^2 \times \mathbf{S}^1)$, where the boundaries of the meridian discs $m_2^{(j)}$, $0 \leq j \leq a-1$, are given by the paths

$$\partial m_2^{(j)} = \langle j, mp + j, 2mp + j, \dots, (q-1)mp + j, qmp + j, (m-1)pq + j, (m-2)pq + j, \dots, j \rangle;$$

- for $r > 0$, $F_3(p, q, r) \cong \mathbf{B}^2 \times \mathbf{S}^1$ where the boundary of the meridian disc m_3 is given by the path

$$\partial m_3 = \langle 0, pq, pq + 1, pq + 2, \dots, -pq - 1, -pq, 0 \rangle;$$

- for the limit case $r = 0$, $F_3(p, q, 0)$ is a collection of pq tetrahedra glued together along common edges, forming a solid torus pinched along pq edges.

Proof. First let us assume that $(p-k)q \geq kq$. By definition we have

$$\begin{aligned} F_1(p, q, r) &= \{(b_i : a_i : b_i : 2pq - 2b_i - a_i + r) \mid 1 \leq i \leq N((p-k)q, kq)\} \\ &= \{d_i \mid 1 \leq i \leq N((p-k)q, kq)\} \end{aligned}$$

where $N((p-k)q, kq)$ denotes the number of steps to compute $\gcd((p-k)q, kq) = q$ using the Euclidean algorithm given by Equation (4.1), and (a_i, b_i) denotes the arguments of the Euclidean algorithm *after* the i -th step (see Section 4.1 for details).

$\partial F_1(p, q, r)$ is contained in d_1 , and by construction d_i can be collapsed onto d_{i+1} whenever each tetrahedron of d_i contains a boundary face of the complex. Hence $F_1(p, q, r)$ can be collapsed onto $d_{N((p-k)q, kq)} = \{(q : q : q : 2pq + r - 3q)\}$. By definition, we have $\gcd(q, r) = b$ and hence

$$\begin{aligned} \gcd(q, 2pq + r - 3q) &= \gcd(q, 2pq + r - 3q) \\ &= \gcd(q, (2p-3)q + r) \\ &= \gcd(q, r) \\ &= b. \end{aligned}$$

It follows that $F_1(p, q, r)$ collapses to b connected components each with $(2pq + r)/b$ vertices and all isomorphic to

$$\{(q/b : q/b : q/b : (2pq + r - 3q)/b)\} \cong \{(1 : 1 : 1 : (2pq + r)/b - 3)\}$$

and thus

$$F_1(p, q, r) \cong \{1, 2, \dots, b\} \times (\mathbf{B}^2 \times \mathbf{S}^1)$$

(see Figure 5.1).

The proof for the case $(p - k)q < kq$ is completely analogous as is the proof that

$$F_2(p, q, r) \cong \{1, 2, \dots, a\} \times (\mathbf{B}^2 \times \mathbf{S}^1)$$

(see Figure 5.1 again). To see that $F_3(p, q, r)$ is a solid torus for $r > 0$ and a collection of tetrahedra glued together along common edges for $r = 0$, just note that it coincides with the last $\lfloor \frac{r}{2} \rfloor + 1$ difference cycles of the boundary complex of the cyclic polytope $\partial C_4(2pq + r)$. For more about how the boundary complex of the cyclic 4-polytope can be decomposed into difference cycles, see [26].

$$\begin{aligned} \partial F_1(p, q, r) &\longrightarrow d_1 \longrightarrow d_2 \longrightarrow \dots \longrightarrow d_{n(mp, p(q-m))} = (p : p : p : n - 3p) \\ \partial F_2(p, q, r) &\longrightarrow d_1 \longrightarrow d_2 \longrightarrow \dots \longrightarrow d_{n(kq, q(p-k))} = (q : q : q : n - 3q) \end{aligned}$$

Figure 5.1: $F_1(p, q, r)$ and $F_2(p, q, r)$ collapsing onto multiple solid tori.

In order to prove that $\partial m_1^{(i)}$, $0 \leq i \leq b - 1$, is the boundary of a meridian disc of $F_1(p, q, r)$ we have to show that $\partial m_1^{(i)} \subset \partial F_1(p, q, r)$ is *i)* closed, *ii)* simple, *iii)* homologous to zero inside $F_1(p, q, r)$, and *iv)* homologically non-trivial in $\partial F_1(p, q, r)$.

Again, let $(p - k)q \geq kq$. The fact that *i)* holds follows immediately from the definition. To see that *ii)* is true assume there is a point of self-intersection, that is, $x \cdot (kq) = y \cdot (pq)$ for integers $0 < x \leq p$ and $0 < y \leq k$. Then

$$\begin{aligned} x \cdot (kq) = y \cdot (pq) &\Leftrightarrow x \cdot (mp - 1) = y \cdot (pq) \\ &\Leftrightarrow x \cdot (mp) - x = y \cdot (pq) \\ &\Leftrightarrow x = p \cdot (xm - yq) \\ &\Leftrightarrow p \mid x \end{aligned}$$

and since $0 < x \leq p$, the only solution is $x = p$, $y = k$, and therefore $\partial m_1^{(i)}$ is simple. To prove *iii)* note that by construction we can homotopically deform $\partial m_1^{(i)}$ over triangles (that is, replace $\langle \dots, v, w, \dots \rangle$ by $\langle \dots, v, u, w, \dots \rangle$ if $\langle u, v, w \rangle$ is a triangle) such that

$$\begin{aligned} \langle i, kq + i, \dots, (p - 1)kq + i, k(pq) + i, (k - 1)pq + i, \dots, i \rangle &\cong \langle i, p + i, \dots, (kq - 1) \cdot p, kq \cdot p, (kq - 1) \cdot p, \dots, i \rangle \\ &\cong 0. \end{aligned}$$

Finally, to prove *iv)* we observe that $\partial m_1^{(i)}$ wraps q/b times around the fundamental domain of the i -th boundary component of $F_1(p, q, r)$ (cf. Figure 5.5) in the horizontal direction and hence cannot be homologous to zero in $\partial F_1(p, q, r)$. All together it follows that $m_1^{(i)}$ is a meridian disc of the i -th connected component of $F_1(p, q, r)$. Again, the proof in the case $(p - k)q < kq$ and the proof for $m_2^{(j)}$, $1 \leq j \leq a - 1$, are completely analogous.

To see that m_3 is a meridian disc of $F_3(p, q, r)$, $r > 0$, see Figure 5.2 where $m_3 \subset F_3(p, q, r)$ is given explicitly. □

Lemma 5.3. *Let \mathcal{B} be a thickened version of the complex $B(p, q, r)$, $r > 0$, such that \mathcal{B} is orientable and all boundary components of $B(p, q, r)$ are disjoint. Then*

$$\mathcal{B} \cong \mathbf{S}^1 \times \mathbb{S}_{\frac{1}{2}(a-1)(b-1)}^{b+a+1},$$

where \mathbb{S}_g^m is the m -punctured orientable surface of genus g .

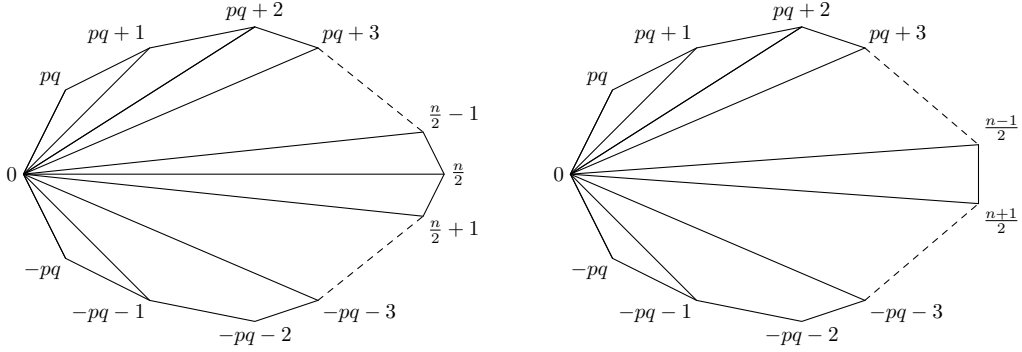


Figure 5.2: The meridian disc of $F_3(p, q, r)$ for $n = 2pq + r$ even (left) and odd (right).

Proof. In essence, we read off the diagrams given in Figures 5.3 and 5.4. The rest of the proof consists of details and bookkeeping.

$B(p, q, r)$ consists of three difference cycles of full length and hence contains $3n = 6pq + 3r$ tetrahedra. These split into n disjoint systems of representatives for the difference cycles of 3 tetrahedra each. One of these systems of representatives is given by

$$\langle 0, 1, mp, pq \rangle, \langle 1, mp, pq, pq + 1 \rangle \text{ and } \langle mp, pq, pq + 1, p(q + m) \rangle.$$

Figure 5.3 illustrates how these n groups of 3 tetrahedra can be stacked onto the fundamental domain of the boundary torus

$$\begin{aligned} \partial F_3(p, q, r) &= \{(1 : pq - 1 : r + pq), (1 : r + pq : pq - 1)\} \\ &= \{\langle 0, 1, pq \rangle, \langle 1, pq, pq + 1 \rangle, \dots\}. \end{aligned}$$

Here two vertically neighbouring groups are glued together along the triangles $\langle pq, pq + 1, p(q + m) \rangle$ and their translates, and the complex $B(p, q, r)$ is obtained by identifying pairs of vertical edges of the resulting complex given in Figure 5.4 where each vertical edge of $\partial F_3(p, q, r)$ (for example $\langle 0, pq \rangle$ and $\langle 1, pq + 1 \rangle$ in the lower left corner) is glued to the unique vertical edge with the corresponding vertex labels not touching the fundamental domain (for example $\langle mp, p(q + m) \rangle$). Note that it already follows from the cyclic symmetry that exactly n of these pairs of vertical edges exist.

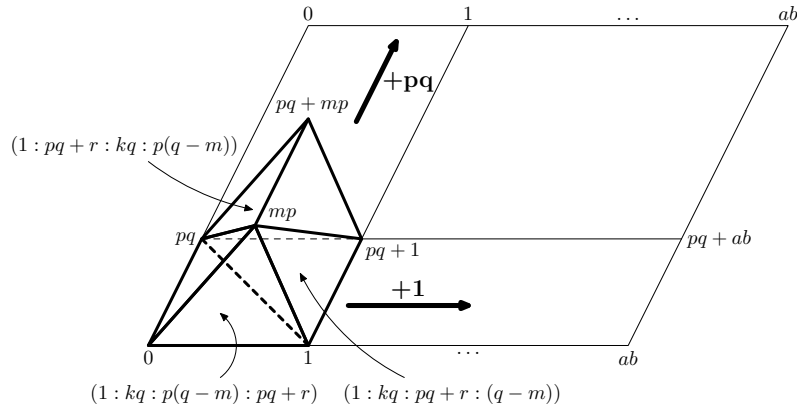


Figure 5.3: System of representatives of the three difference cycles of $B(p, q, r)$.

This construction together with Lemma 5.2 gives rise to a complex with $b + a + 1$ boundary tori where all the boundary tori run vertically relative to the fundamental domain given in Figure 5.4. A more schematic drawing of $B(p, q, r)$ together with its boundary tori is given in Figure 5.5.

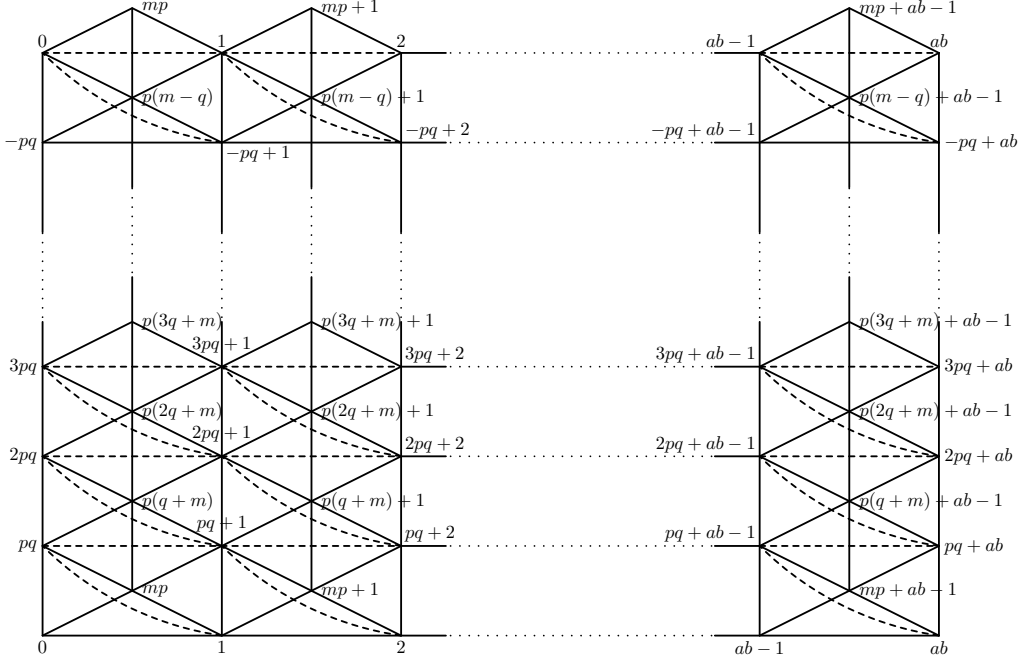


Figure 5.4: $B(p, q, r)$ after gluing tetrahedra along common boundary faces. Triangles at the top (e.g., $\langle 0, 1, mp \rangle$) are glued to the bottom ones. Edges with equally labelled endpoints are identified.

This already tells us that $B(p, q, r)$ is the Cartesian product of a circle with S , where S is a closed surface minus $b + a + 1$ discs. S runs horizontally relative to the fundamental domain (i.e., S meets $\partial F_3(p, q, r)$ in a curve of the same homotopy class as the horizontal line in the fundamental domain, plus a necessary vertical shift at the right hand side to close it); and the circle runs vertically. In order to see what S looks like, we must pay attention to how exactly the vertical edges of the complex are glued together. A basic observation exploiting the cyclic symmetry of the complex tells us that every vertical line in Figure 5.4 of $B(p, q, r)$ contains a single vertex β such that $0 \leq \beta < ab$. For the vertical lines touching the fundamental domain of $\partial F_3(p, q, r)$ these are at the very bottom except for at the rightmost line where vertex 0 is shifted by α , where α describes how the vertical boundary parts of the fundamental domain of $F_3(p, q, r)$ are shifted in order to be glued together to build a torus. For the other vertical lines the unique vertex label β , $0 \leq \beta < ab$, is shifted by $\frac{n}{ab} - \gamma$ or $\frac{n}{ab} - \gamma + \alpha$, where γ describes the vertical distance (modulo $\frac{n}{ab}$) of β and vertex mp . Figure 5.6 shows the cut through $B(p, q, r)$ containing all of these vertices and thus resulting in a simple representation of the base surface S .

Note that S after identifying vertices with equal labels contains exactly $b + a + 1$ edge disjoint boundary circles such that each belongs to a unique connected component of $F_i(p, q, r)$, $1 \leq i \leq 3$. Each of these boundary circles can be given an orientation such that each of their edges is oriented clockwise in the drawing of S given in Figure 5.6. It follows that S , and hence $B(p, q, r)$, can be thickened to give a bounded 3-manifold \mathcal{B} homeomorphic to the Cartesian product of the circle with an oriented surface with $b + a + 1$ punctures. Note that S has ab vertices, $3ab$ edges and ab triangles, hence Euler characteristic $\chi(S) = -ab$, and we have $S \cong \mathbb{S}_{\frac{1}{2}(a-1)(b-1)}^{b+a+1}$. \square

Lemma 5.4. *Let \mathcal{B} be a thickened version of $B(p, q, 0) \cup F_3(p, q, 0)$, with a slightly thickened version of $F_3(p, q, 0)$ drilled out, such that \mathcal{B} is orientable and all boundary components of $B(p, q, 0)$ are disjoint. Then*

$$\mathcal{B} \cong \mathbf{S}^1 \times \mathbb{S}_{\frac{1}{2}(p-1)(q-1)}^{p+q+1},$$

where \mathbb{S}_g^m is the m -punctured orientable surface of genus g .

Proof. The proof is largely analogous to the proof of Lemma 5.3 with some minor changes.

Since $r = 0$ we have $a = \gcd(p, 0) = p$ and $b = \gcd(q, 0) = q$. Hence Figure 5.4 has only two rows (note that $pq + ab = 2pq = 0$), where the top-row is identified with the bottom row by folding them up, leaving pq tetrahedron-shaped holes with boundaries of type

$$\langle \langle \ell, \ell + 1, pq + \ell \rangle, \langle \ell, \ell + 1, pq + 1 + \ell \rangle, \langle 0, pq + \ell, pq + 1 + \ell \rangle, \langle 1, pq + \ell, pq + 1 + \ell \rangle \rangle$$

for $0 \leq \ell \leq pq - 1$ which, in $M(p, q, 0)$, are filled with the pq tetrahedra of $F_3(p, q, 0) = \{(1 : pq - 1 : 1 : pq - 1)\}$. Drilling out a slightly thickened version of $F_3(p, q, 0)$ leaves us with a torus boundary component on the bottom of Figure 5.4 (as long as $B(p, q, 0)$ has been sufficiently thickened before near the edges $\langle \ell, pq + \ell \rangle$, $0 \leq \ell \leq pq - 1$).

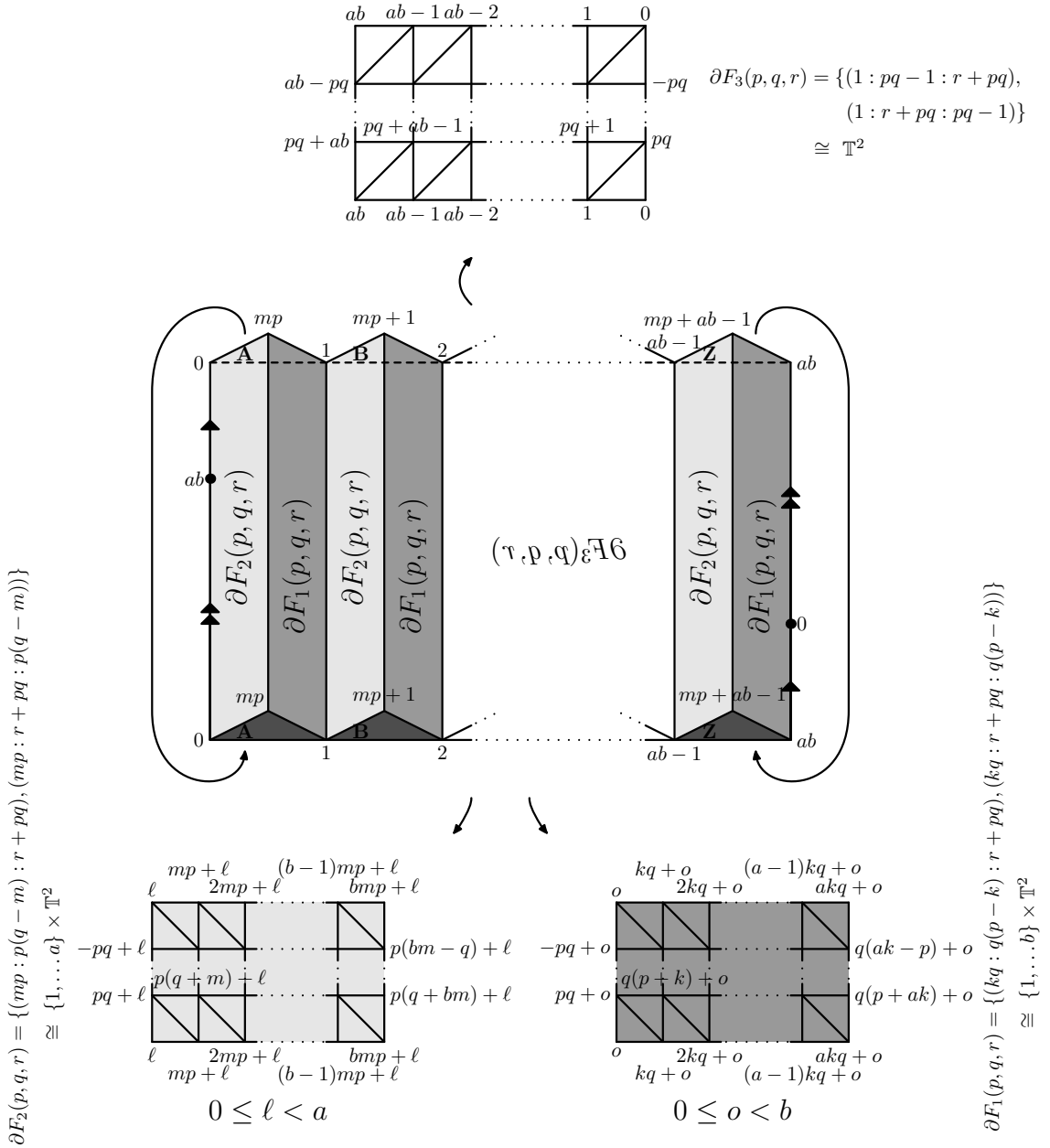


Figure 5.5: The $(a + b + 1)$ boundary tori of $B(p, q, r)$.

Hence we get a space with $a + b + 1 = p + q + 1$ boundary tori. A two-row version of Figure 5.5 shows the complex before thickening and drilling. As in the proof of Lemma 5.3 the boundary tori run vertically relative to the fundamental domain.

This tells us that \mathcal{B} is the Cartesian product of a surface minus $p + q + 1$ disjoint discs S with a circle, where S is running horizontally relative to the fundamental domain. The hypothesis now follows analogously with the shifts $\alpha = -1$, $\beta = mp$ and $\gamma = 0$. \square

Lemma 5.5. *Relative to the fundamental domain and base orbifold chosen in Figure 5.6, the types of the exceptional fibres for $r > 0$ are*

- $(-\frac{p}{a}, \frac{ab}{n}(p\gamma - \frac{p\alpha(\beta-1)}{ab} - k))$ for the b exceptional fibres of $F_1(p, q, r)$,
- $(\frac{q}{b}, \frac{ab}{n}(q\gamma - \frac{q\alpha\beta}{ab} - m))$ for the a exceptional fibres of $F_2(p, q, r)$, and
- $(-\frac{r}{ab}, \frac{ab}{n}(2 - \frac{\alpha r}{ab}))$ for the exceptional fibre of $F_3(p, q, r)$,

where

$$\alpha = -\left(\left(\frac{pq}{ab}\right)^{-1} \mod \frac{n}{ab}\right), \quad \beta = mp \mod ab \quad \text{and} \quad \gamma = \left(\frac{pq}{mp - \beta}\right)^{-1} \mod \frac{n}{ab}$$

are the shifts defining the identifications in $B(p, q, r)$ as shown in Figure 5.6.

For $r = 0$ we get q fibres of type $(-1, 0)$, p fibres of type $(1, 0)$, and one fibre of type $(0, 1)$.

Proof. Again, we proof the statement by looking at Figures 5.5 and 5.6.

An exceptional fibre is of type (a, b) if the meridian disc of its solid torus neighbourhood is glued to a closed curve in the corresponding boundary torus of $M(p, q, r)$ which wraps a times around the torus in the direction of S (this is referred to as *the horizontal direction*) and b times in the direction of the fibres (*the vertical direction*).

In order to determine the exact types of exceptional fibres in $M(p, q, r)$ we have to specify exactly how the vertical lines in the fundamental domain of $B(p, q, r)$ (as shown for example in Figure 5.5) are identified. First of all the top boundary $\langle 0, 1, \dots, ab \rangle$ is identified with the bottom boundary $\langle 0, 1, \dots, ab \rangle$ without any shift and as indicated by the vertex labels. The left boundary $\langle 0, pq, 2pq, \dots, 0 \rangle$ is identified with the right boundary $\langle ab, pq + ab, 2pq + ab, \dots, ab \rangle$ by shifting the right boundary *down* by α rows (cf. 5.6). Now the vertical lines of type $\langle mp, pq + mp, 2pq + mp, \dots, mp \rangle \subset \partial F_i(p, q, r)$, $1 \leq i \leq 2$, are glued to their counterparts in $\partial F_3(p, q, r)$ by shifting them β columns to the *right* and γ rows *up* (cf. 5.6).

Finally, we assign a positive orientation to all fibres that run from the bottom to the top of the fundamental domain and to all horizontal paths which run from the left to the right on the front (boundary components of $F_1(p, q, r)$ and $F_2(p, q, r)$) and hence from the right to the left in the back ($\partial F_3(p, q, r)$) of $B(p, q, r)$.

Following this framework note that the boundary curves of the meridian discs $\partial m_1^{(i)}$, $0 \leq i \leq b - 1$, have exactly length p in the horizontal direction, and that the fundamental domains of the corresponding boundary tori (cf. Figure 5.5) have exactly a columns. Furthermore, $\partial m_1^{(i)}$ runs from the right to the left and hence $\partial m_1^{(i)}$ wraps around the fundamental domain of $B(p, q, r)$ exactly $-p/a$ times in the horizontal direction. Using the same reasoning we can see that $\partial m_2^{(j)}$, $0 \leq j \leq a - 1$, wraps around the fundamental domain of $B(p, q, r)$ exactly q/b times in the horizontal direction and ∂m_3 exactly $-r/(ab)$ times.

To determine how often the boundary curves of the meridian discs wrap around the fundamental domain in the vertical direction, we have to carefully take into account the shifts α , β and γ of the identifications of the vertical lines in $B(p, q, r)$ (see Figure 5.6 for details).

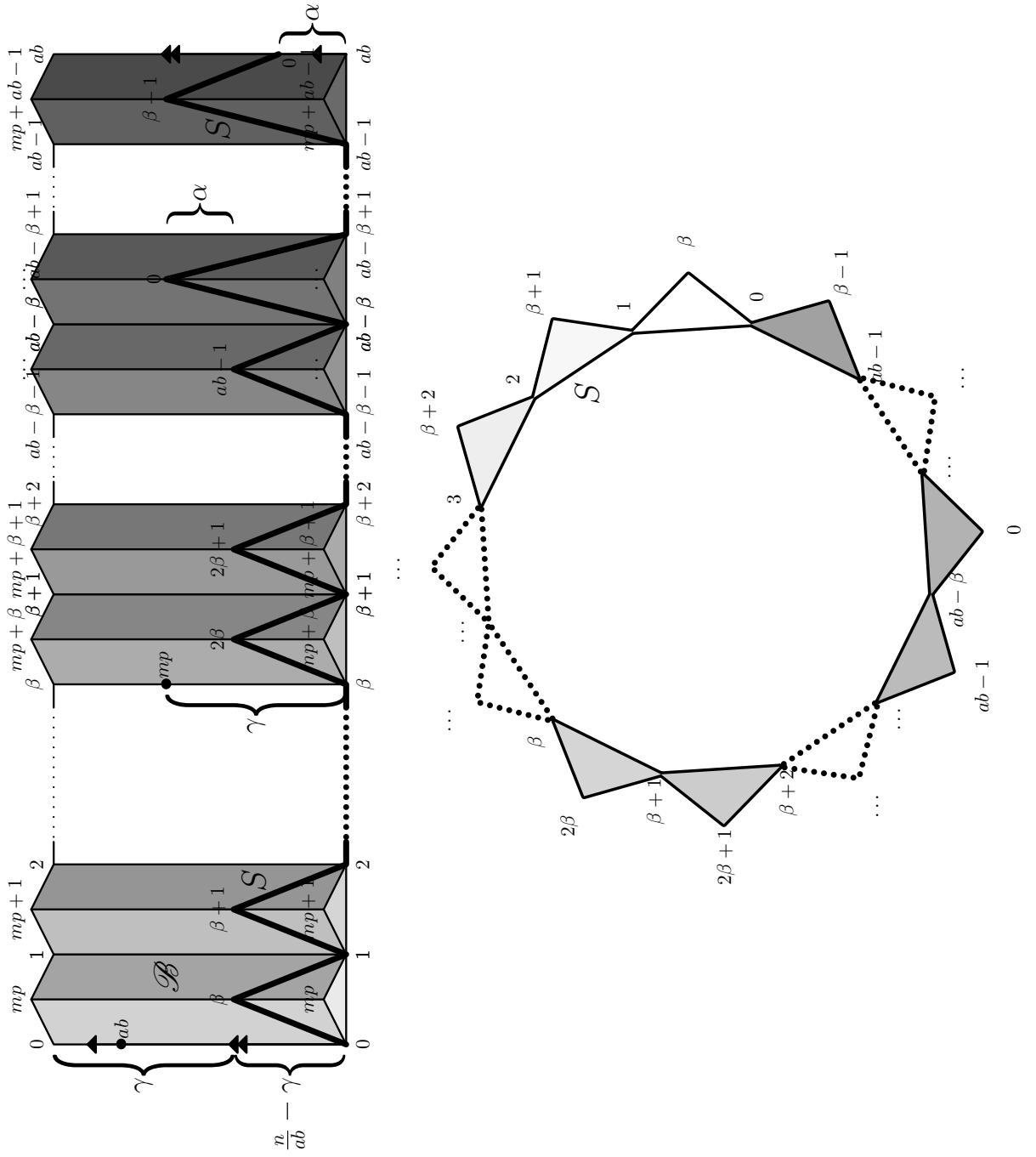


Figure 5.6: Position of the base surface S in the trivial \mathbf{S}^1 -bundle \mathcal{B} .

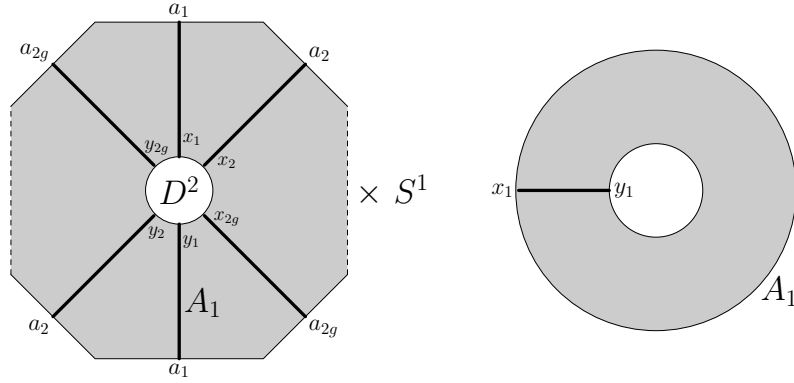


Figure 5.7: On the left: the trivial circle bundle over a punctured orientable genus g surface $M \setminus T_1$. On the right: a fibre cross interval of $M \setminus T_1$.

The boundary curves of the meridian discs $\partial m_1^{(i)}$, $0 \leq i \leq b-1$, have length $-k$ in the vertical direction and are shifted p times in the positive vertical direction by γ rows. In addition to this, $\partial m_1^{(i)}$ runs p times half-columns in the negative horizontal direction followed by a shift of $\beta - 1/2$ columns in the positive horizontal direction. This results in p times a horizontal shift of $\beta - 1$ columns and for each horizontal shift of ab columns in positive direction we have to add another vertical shift of α rows in the negative direction. In other words, $\partial m_1^{(i)}$ is shifted in the positive vertical direction by exactly

$$p \cdot \gamma - \frac{p\alpha(\beta-1)}{ab} - k$$

rows. The fact that all fundamental domains consist of $\frac{n}{ab}$ rows then proves the result.

The vertical shifts of $\partial m_2^{(j)}$, $0 \leq j \leq a-1$, and ∂m_3 are computed in an analogous fashion. All together the exceptional fibres are as stated.

For the case $r = 0$, note that $pq/(ab) = 1$, $n/(ab) = 2$, and $mp < pq$. Hence we get $\alpha = -1$, $\beta = mp$, $\gamma = 0$, $p+q$ fibres of type $(\pm 1, 0)$ and one fibre of type $(0, 1)$. \square

Lemma 5.6. *Let $M = \mathbb{S}_g \times \mathbf{S}^1$ be the trivial \mathbf{S}^1 -bundle over the orientable surface of genus g , and let M' be obtained from M by performing surgery of type $(0, 1)$ along the \mathbf{S}^1 -fibre. Then $M' \cong (\mathbf{S}^2 \times \mathbf{S}^1)^{\#2g}$.*

Proof. We start by representing M as a product of a $4g$ -gon with opposite edges identified and a circle \mathbf{S}^1 . Let $T_1 \subset M$ be a solid torus $T_1 = \mathbf{D}^2 \times \mathbf{S}^1$ inside M where the first factor \mathbf{D}^2 is a disc inside the $4g$ -gon and the second factor \mathbf{S}^1 is a copy of the fibre of M ; see Figure 5.7 for a picture of $M \setminus T_1$. Now let $T_2 = \mathbf{S}^1 \times \mathbf{D}^2$ be a solid torus, and let $M' = (M \setminus T_1) \cup T_2$ such that the \mathbf{S}^1 -factor of T_2 is glued to the boundary of the \mathbf{D}^2 -factor of T_1 , and the boundary of the \mathbf{D}^2 -factor of T_2 is glued to a copy of the \mathbf{S}^1 -factor of T_1 on the boundary of $M \setminus T_1$. In other words, M' is obtained by performing surgery in M of type $(0, 1)$ along the fibre. Denote by $i : T_2 \rightarrow M'$ the embedding of T_2 into M' defined by this surgery.

It follows that there exist $4g$ disjoint disks of type $\{x_i\} \times \mathbf{D}^2$ and $\{y_i\} \times \mathbf{D}^2$, $1 \leq i \leq 2g$, with x_i, y_i as indicated in Figure 5.7, which close off the $2g$ disjoint annuli $A_i = [x_i, y_i] \times \mathbf{S}^1$ inside M' , yielding $2g$ (simultaneously) non-separating disjoint two-spheres inside M' . To see why the spheres are non-separating, note that all corners of the $4g$ -gon are identified in M' and every piece of M' after cutting out the spheres is still connected to one of the corners.

Now cutting along all $2g$ of these 2-spheres yields $2g$ pieces of type $(\mathbf{S}^2 \times \mathbf{S}^1) \setminus \mathbf{D}^3$ and a 3-sphere with $2g$ punctures. Hence M' is homeomorphic to a connected sum of type $(\mathbf{S}^2 \times \mathbf{S}^1)^{\#2g}$. \square

With these building blocks in mind we can now finish the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $r > 0$. First of all, by Lemma 5.1 the topological type of $M(p, q, r)$ as stated in Theorem 1.2 is unique and thus well-defined.

Now, by Lemma 5.3 we know that a thickened version \mathcal{B} of $B(p, q, r)$ is homeomorphic to $\mathbf{S}^1 \times \mathbb{S}_{\frac{1}{2}(a-1)(b-1)}^{b+a+1}$. We construct \mathcal{B} by gluing a small neighbourhood of the boundary of each of the $b + a + 1$ connected components of $F_i(p, q, r)$, $1 \leq i \leq 3$, to $B(p, q, r)$. This results in the space $\mathbf{S}^1 \times \mathbb{S}_{\frac{1}{2}(a-1)(b-1)}^{b+a+1}$ as, by Lemma 5.2 each of these components is a solid torus.

By Lemma 5.5, the boundary curves of the meridian discs of these solid tori are of type

- $(-\frac{p}{a}, b_1)$ for the b exceptional fibres of $F_1(p, q, r)$ for $b_1 = \frac{ab}{n}(p\gamma - \frac{p\alpha(\beta-1)}{ab} - k)$,
- $(\frac{q}{b}, b_2)$ for the a exceptional fibres of $F_2(p, q, r)$ for $b_2 = \frac{ab}{n}(q\gamma - \frac{q\alpha\beta}{ab} - m)$, and
- $(-\frac{r}{ab}, b_3)$ for the exceptional fibre of $F_3(p, q, r)$ for $b_3 = \frac{ab}{n}(2 - \frac{\alpha r}{ab})$.

Note that the number of exceptional fibres is correct and changing the signs of the indices in the horizontal direction of all exceptional fibres simultaneously results in the desired values p/a , $-q/b$ and $r/(ab)$ but only reverses the orientation of the Seifert fibration. Thus it remains to show that

$$\begin{aligned} \left(\frac{b_1}{p} - \frac{b_2}{q} + \frac{b_3}{r}\right) \frac{pqr}{ab} &= \frac{qrb_1 - prb_2 + pqb_3}{ab} \\ &= \frac{qr(\frac{ab}{n}(p\gamma - \frac{p\alpha(\beta-1)}{ab} - k)) - pr(\frac{ab}{n}(q\gamma - \frac{q\alpha\beta}{ab} - m)) + pq(\frac{ab}{n}(2 - \frac{\alpha r}{ab}))}{ab} \\ &= \frac{qr(p\gamma - \frac{p\alpha(\beta-1)}{ab} - k) - pr(q\gamma - \frac{q\alpha\beta}{ab} - m) + pq(2 - \frac{\alpha r}{ab})}{ab} \\ &= \frac{pqr(\gamma - \gamma + \frac{q\alpha\beta}{ab} - \frac{q\alpha\beta}{ab} + \frac{\alpha}{ab} - \frac{\alpha}{ab}) + rmp - rkq + 2pq}{n} \\ &= \frac{r(mp - kq) + 2pq}{n} \\ &= 1, \end{aligned}$$

which proves Theorem 1.2 for $r > 0$.

Let $r = 0$. By Lemma 5.4, $M(p, q, 0)$ can be obtained from the Cartesian product $\mathbf{S}^1 \times \mathbb{S}_g^0$, $g = \frac{1}{2}(p-1)(q-1)$, by performing $p + q + 1$ surgeries along the \mathbf{S}^1 component. By Lemma 5.5, all but one of these surgeries are of trivial type $(\pm 1, 0)$ and do not change the topology. Hence $M(p, q, 0)$ is obtained from $\mathbf{S}^1 \times \mathbb{S}_g^0$ by a single surgery along \mathbf{S}^1 of type $(0, 1)$, i.e., by drilling out a solid torus along the \mathbf{S}^1 component and gluing it back in with meridian and longitude interchanged. By Lemma 5.6 we thus have

$$M(p, q, 0) = (\mathbf{S}^2 \times \mathbf{S}^1)^{\#(p-1)(q-1)}.$$

□

6 Acknowledgements

This work was supported by the Australian Research Council under the Discovery Projects funding scheme, project DP1094516. Furthermore, the authors want to thank Wolfgang Kühnel and the anonymous referee for many valuable comments.

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